Critical Dimension, Variational Stability in the Niemeier Landscape, and the Lorentzian Geometry of the Cannonball Identity

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Abstract

The identity

$$1^2 + 2^2 + \cdots + 24^2 = 70^2$$

has a natural realization in Conway's Lorentzian construction of the Leech lattice. In the even unimodular Lorentzian lattice $\mathrm{II}_{25,1}$, a distinguished primitive null vector can be represented, in a specific diagonal coordinate realization, by

$$(0, 1, 2, \ldots, 24 \mid 70).$$

Its Lorentzian nullity is therefore equivalent to the classical cannonball identity.

The appearance of the number 24 invites comparison with critical bosonic string theory, whose BRST consistency condition fixes the spacetime dimension to 26, leaving 24 transverse directions. We emphasize, however, that criticality alone does not select the Leech lattice. If one further models the relevant Euclidean sector by a positive-definite even unimodular lattice of rank 24, the resulting finite classification is the Niemeier landscape consisting of twenty-three rooted Niemeier lattices together with the unique rootless Leech lattice.

We separate six logically distinct structures. First, worldsheet criticality fixes the transverse dimension to 24, but not the lattice. Second, the Niemeier classification and the associated root-system data provide an exact algebraic diagnostic: rootlessness is equivalent to vanishing Weyl-vector contribution and uniquely characterizes the Leech lattice. Third, Conway's Lorentzian realization associates primitive null directions with Niemeier cusps; Lorentzian nullity is therefore universal across the landscape and is not by itself a Leech selector. Fourth, a model-dependent variational problem for minimal spherical shells with the Riesz-type potential $f(u) = u^{-2}$ yields exact positive-semidefinite Hessians for the A_1^{24} root shell and the Leech minimal shell, while the remaining rooted shells exhibit computationally certified unstable directions. Fifth, global universal optimality may be invoked separately using completely monotonic potentials for which the 24-dimensional lattice energy is well-defined, thereby distinguishing the Leech lattice globally. Finally, Conway's specific diagonal coordinate realization converts the Leech null vector into the cannonball identity.

The resulting hierarchy is conditional: conventional bosonic string criticality sets the exceptional dimension 24, but does not by itself predict either the Leech lattice or the integer 70.

Contents

1 Introduction

In 1875, Édouard Lucas posed the classical cannonball problem: determine the square numbers that can be represented as sums of consecutive squares starting from 1. The non-trivial integer solution is

$$\sum_{k=1}^{24} k^2 = \frac{24 \cdot 25 \cdot 49}{6} = 4900 = 70^2. \tag{1}$$

The classical Diophantine problem was solved by Watson [18].

The same identity has a natural geometric realization in Conway's treatment of the Leech lattice. In a suitable diagonal realization of the even Lorentzian unimodular lattice $\mathrm{II}_{25,1}$, one may represent a distinguished primitive null vector by

$$w = (0, 1, 2, \ldots, 24 \mid 70). \tag{2}$$

Its nullity is precisely

$$w^2 = \sum_{k=1}^{24} k^2 - 70^2 = 0.$$

This vector occurs in the Lorentzian realization in which the Leech lattice is recovered from an appropriate orthogonal quotient.

The number 24 also occurs naturally in critical bosonic string theory. BRST consistency fixes the matter central charge to 26, and after passage to light-cone variables the physical transverse sector has dimension

$$26 - 2 = 24.$$

This coincidence raises a natural question:

> To what extent can the critical dimension, the Niemeier landscape, Lorentzian geometry, and variational lattice theory be connected into a single mathematical framework leading to the Leech lattice and the cannonball identity?

The answer requires careful distinction among several structures. Criticality determines a dimension; it does not, without additional assumptions, determine a particular positive-definite even unimodular lattice. If one imposes such a lattice structure on the 24-dimensional Euclidean sector, the Niemeier classification supplies a finite landscape. Borcherds-type automorphic constructions provide root-system and Weyl-vector data. Conway's Lorentzian realization relates these lattice structures to primitive null vectors. Separately, variational energy functionals can be used to define model-dependent notions of local stability and global minimization.

The purpose of this paper is therefore not to assert that critical string theory alone predicts the integer 70. Rather, we formulate and analyze a conditional mathematical architecture connecting these structures.

1.1 Logical architecture

The central distinction is summarized schematically by

$$\begin{array}{rcl}
\textbf{Stage 1: Criticality} & \Longrightarrow & D = 26, \quad d_\perp = 24 \\
\textbf{Stage 2: Lattice assumption} & \Longrightarrow & K \text{ even, unimodular, positive-definite, } \operatorname{rk} K = 24 \\
\textbf{Stage 3: Classification} & \Longrightarrow & \mathcal{N}_{24} = \{23 \text{ rooted Niemeier lattices}\} \cup \{\Lambda_{24}\} \\
\textbf{Stage 4: Algebraic diagnostic} & \Longrightarrow & \rho_K = 0 \iff K \cong \Lambda_{24} \\
\textbf{Stage 5: Lorentzian geometry} & \Longrightarrow & \text{primitive null vectors associated with Niemeier cusps} \\
\textbf{Stage 6: Variational filtration} & \Longrightarrow & \text{model-dependent Hessian stability on minimal shells} \\
\textbf{Stage 7: Global optimality} & \Longrightarrow & \Lambda_{24} \text{ under suitable admissible energies} \\
\textbf{Stage 8: Conway coordinates} & \Longrightarrow & 1^2 + \cdots + 24^2 = 70^2.
\end{array} \tag{3}$$

Stages 1 and 3 concern established structures in string theory and lattice theory, while Stage 2 is an additional modeling assumption. Stage 4 uses the root-system structure of the Niemeier classification. Stage 5 concerns Lorentzian lattice geometry. Stage 6 contains model-dependent spectral calculations on minimal spherical shells. Stage 7 invokes established universal-optimality theorems under their stated hypotheses. Stage 8 requires a specific coordinate realization of the Leech cusp; Lorentzian nullity alone does not determine the integer sequence $(0, 1, \ldots, 24 \mid 70)$.

2 Status of Results and Scope

For clarity, we classify the mathematical content of the paper into three categories.

2.1 Established background

The following ingredients are classical or established independently of the present work:

- BRST criticality of the bosonic string at $D = 26$;
- the classification of positive-definite even unimodular lattices in rank 24;
- uniqueness of the rootless Leech lattice in that classification;
- the Lorentzian realization of the Leech lattice inside $\mathrm{II}_{25,1}$;
- the Weyl-vector identities for simply-laced Niemeier root systems;
- universal optimality of the Leech lattice in dimension 24 for completely monotonic functions of squared distance, under the hypotheses of the corresponding theorem;
- the classical cannonball identity.

2.2 Results used as computational input

The exact Hessian spectra, characteristic-polynomial factorizations, representation-theoretic multiplicities, and rational trace identities reported below are results of the accompanying computational program.

Where an equality is obtained by exact rational arithmetic, we describe it as an exact computation. Numerical residuals or eigenvalue comparisons with tolerances such as 10^{-14} are explicitly identified as floating-point checks.

2.3 Interpretive limitation

No claim is made that the Riesz potential

$$f(u) = u^{-2}$$

is a fundamental interaction of perturbative bosonic string theory. The variational construction is instead an explicit mathematical model that provides a possible filtration of the finite Niemeier landscape.

Likewise, no claim is made that the number 70 follows from BRST nilpotency alone. The number 70 enters through the particular Conway Lorentzian coordinate realization of the Leech lattice.

Finally, the local shell-stability calculation and the global lattice-energy comparison are treated as distinct variational problems. In particular, the local potential u^{-2} is not used as an absolutely convergent infinite-lattice energy in dimension 24.

3 Criticality and the Niemeier Landscape

In covariant quantization of the bosonic string in flat spacetime $\mathbb{R}^{D-1,1}$, the matter sector contains D free scalar fields $X^\mu(z, \bar{z})$ and therefore has central charge

$$c_{\text{matter}} = D.$$

Gauge fixing introduces the (b, c) ghost system with

$$c_{\text{ghost}} = -26.$$

BRST consistency requires cancellation of the total conformal anomaly:

$$Q_{\text{BRST}}^2 = 0 \quad \Longrightarrow \quad c_{\text{matter}} + c_{\text{ghost}} = 0, \tag{4}$$

and hence

$$D = 26. \tag{5}$$

After passage to light-cone variables, the physical transverse sector has dimension

$$D - 2 = 24.$$

At this point an additional lattice assumption is required. Suppose that the relevant Euclidean sector is modeled by a positive-definite even unimodular lattice K of rank 24. The Niemeier classification then gives exactly twenty-four such lattices up to isometry: twenty-three rooted lattices and the unique rootless Leech lattice [13, 9].

We write

$$\mathcal{N}_{24} = \{N(R)\} \cup \{\Lambda_{24}\}.$$

Thus the logical implication is

$$D = 26 \quad \Longrightarrow \quad d_\perp = 24, \tag{6}$$

followed only after the additional lattice assumption by

$$d_\perp = 24 \quad + \quad K \text{ even unimodular} \quad \Longrightarrow \quad K \in \mathcal{N}_{24}. \tag{7}$$

Criticality alone therefore does not select a member of the Niemeier classification.

3.1 Lattice vertex operator algebras

For an even positive-definite lattice K, the associated lattice vertex operator algebra V_K has central charge

$$c = \mathrm{rk}\, K.$$

For a unimodular lattice, V_K is holomorphic. In rank 24, all the Niemeier lattice VOAs lie in the same central-charge class $c = 24$.

Their weight-one spaces distinguish rooted and rootless cases:

$$(V_K)_1 \cong \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(K)} \mathbb{C}e^{\alpha}, \tag{8}$$

where $\mathfrak{h}$ is the 24-dimensional Cartan subalgebra and

$$\Phi(K) = \{\alpha \in K : \alpha^2 = 2\}$$

is the root system.

For a rooted Niemeier lattice, the root system is a direct sum of simply-laced irreducible components sharing a common Coxeter number h. The corresponding weight-one Lie algebra has dimension

$$\dim(V_K)_1 = 24(h+1). \tag{9}$$

For the Leech lattice there are no norm-2 vectors and hence

$$(V_{\Lambda_{24}})_1 \cong \mathbb{C}^{24}. \tag{10}$$

Thus equality of central charge does not distinguish the Leech lattice from the rooted Niemeier lattices.

4 The Borcherds Lift and the Weyl-Vector Diagnostic

Consider the weakly holomorphic modular form

$$f(\tau) = \frac{1}{\Delta(\tau)} = q^{-1} + 24 + 324q + 3200q^2 + \cdots, \qquad q = e^{2\pi i \tau}. \tag{11}$$

Write

$$f(\tau) = \sum_{n=-1}^{\infty} c(n) q^n.$$

Then

$$c(-1) = 1, \qquad c(n) = 0 \quad (n \leq -2).$$

Borcherds' singular theta-lift construction produces automorphic forms associated with orthogonal lattices of Lorentzian signature [2, 3]. At a Niemeier cusp, the associated product expansion encodes the root system through the relevant Fourier coefficients of the input modular form.

For the input $1/\Delta$, the unique negative-index coefficient is $c(-1) = 1$, so norm-2 vectors are singled out in the corresponding product data. With the standard normalization of the Weyl-vector contribution, the Euclidean component agrees with the ordinary Weyl vector of the Niemeier root system:

$$\rho_K = \frac{1}{2} \sum_{\alpha \in \Phi^+(K)} \alpha = \rho_R. \tag{12}$$

Proposition 4.1 (Rootlessness diagnostic)**.** *Within the rank-24 Niemeier classification,*

$$\rho_K = 0 \quad\Longleftrightarrow\quad \Phi(K) = \varnothing \quad\Longleftrightarrow\quad K \cong \Lambda_{24}.$$

Proof. If K is rootless, then $\Phi(K) = \varnothing$ and the sum defining ρ_K is empty, so $\rho_K = 0$.

Conversely, suppose that $\Phi(K)$ is nonempty. It is a finite positive-definite root system. Choosing a positive system, its Weyl vector is a strictly positive linear combination of the corresponding fundamental weights, and therefore is nonzero. Hence $\rho_K = 0$ implies $\Phi(K) = \varnothing$. The uniqueness of the rootless rank-24 even unimodular lattice then gives $K \cong \Lambda_{24}$. □

This proposition is an exact algebraic diagnostic. It is not, by itself, a dynamical selection theorem: it characterizes the Leech lattice once rootlessness has been supplied or otherwise established.

5 Conway's Lorentzian Geometry and Universal Nullity

Let

$$\mathrm{II}_{25,1} \cong K \oplus \mathrm{II}_{1,1},$$

where $\mathrm{II}_{1,1}$ is spanned by isotropic vectors e, f satisfying

$$e^2 = f^2 = 0, \qquad \langle e, f\rangle = -1.$$

For a vector

$$v = x + me + nf,$$

with $x \in K$ and $m, n \in \mathbb{Z}$, the Lorentzian norm is

$$v^2 = x^2 - 2mn. \tag{13}$$

Let R be the root system of a rooted Niemeier lattice, with common Coxeter number h. For an irreducible simply-laced component of rank r_i, the Freudenthal–de Vries strange formula gives

$$\|\rho_i\|^2 = \frac{h \dim \mathfrak{g}_i}{12} = \frac{h\, r_i(h+1)}{12}.$$

Since the total rank is $\sum_i r_i = 24$,

$$\rho_R^2 = \frac{h(h+1)}{12} \sum_i r_i = 2h(h+1). \tag{14}$$

Define the Conway Weyl vector

$$w_R = \rho_R + (h+1)e + hf. \tag{15}$$

Then

$$w_R^2 = \rho_R^2 - 2h(h+1) = 2h(h+1) - 2h(h+1) = 0. \tag{16}$$

Theorem 5.1 (Universal Conway nullity)**.** *For every rooted rank-24 Niemeier lattice, the vector w_R defined by Eq. (15) is Lorentzian null.*

Proof. The computation above gives

$$w_R^2 = 0$$

for every rooted Niemeier lattice, using only the common Coxeter-number property and the strange formula. □

Corollary 5.2. *Lorentzian nullity in* $\mathrm{II}_{25,1}$ *does not, by itself, distinguish the Leech lattice from the rooted Niemeier lattices.*

Remark 5.3 (Clarification regarding the rootless case)**.** There is no Coxeter number associated with the rootless Leech lattice. One may formally extend Eq. (15) by assigning $h = 0$ and $\rho_\Lambda = 0$, which yields

$$w_\Lambda = e.$$

This is only a convenient formal notation and is not a statement of Lie-theoretic structure. More importantly, the equation $w_\Lambda = e$ expresses the Leech cusp as a null direction in the abstract decomposition $K \oplus \mathrm{II}_{1,1}$; it does not by itself fix a diagonal coordinate basis in which the coordinates $(0, 1, \ldots, 24 \mid 70)$ appear.

6 A Variational Filtration on Minimal Spherical Shells

We now introduce a model-dependent variational problem on spherical shells.

Let

$$X = \{x_1, \ldots, x_N\} \subset S_R^{23}$$

be a finite spherical configuration of radius R. For the pair potential

$$f(u) = u^{-2},$$

where

$$u = \|x_i - x_j\|^2$$

is the squared chordal distance, define

$$E(X) = \frac{1}{2} \sum_{i \neq j} f(u_{ij}). \tag{17}$$

The configuration space is

$$\mathcal{M} = (S_R^{23})^N,$$

with tangent dimension

$$\dim T\mathcal{M} = 23N.$$

At a critical configuration, the Riemannian Hessian gives the second variation. A negative eigenvalue produces a direction of negative second variation and hence a local saddle instability for the specified energy functional.

6.1 Single-particle stiffness

For a rooted Niemeier root shell normalized by $R^2 = 2$, the roots have pairwise inner products

$$2,\ 1,\ 0,\ -1,\ -2.$$

For a fixed root, the corresponding nontrivial squared chordal distances are

$$u = 2,\ 4,\ 6,\ 8.$$

The multiplicities are

$$n_1 = 2(h-2), \qquad n_0 = 20h + 6, \qquad n_{-1} = 2(h-2), \qquad n_{-2} = 1.$$

For

$$f'(u) = -2u^{-3}, \qquad f''(u) = 6u^{-4},$$

substitution into the single-particle curvature expression gives the following formula.

Proposition 6.1 (Single-particle stiffness). *For the rooted Niemeier root shell normalized by $R^2 = 2$ and the potential $f(u) = u^{-2}$, the single-particle stiffness is*

$$\lambda_S(h) = \frac{2155 - 416h}{3456}. \tag{18}$$

Consequently,

$$\lambda_S(h) < 0 \qquad \textit{for all integers } h \geq 6.$$

Remark 6.2. This proposition concerns the chosen Riesz functional and normalization. It is not a stability theorem for Niemeier lattices under arbitrary interactions.

6.2 Collective stability and the $23 = 1 + 22$ breakdown

The single-particle stiffness is the diagonal block of the collective Hessian; off-diagonal coupling terms between distinct particles must also be included.

For rooted shells with $h \geq 6$, Eq. (18) supplies a negative diagonal contribution. For

$$h = 3, 4, 5,$$

the sign of the single-particle contribution alone does not determine the collective spectrum, so the full Hessian must be diagonalized.

The computational program associated with this work yields the following lowest eigenvalues for representative rooted shells:

$$\lambda_{\text{ground}}(A_2^{12}) = -\frac{17}{24}, \tag{19}$$

$$\lambda_{\text{ground}}(A_3^{8}) \approx -0.828704, \tag{20}$$

and

$$\lambda_{\text{ground}}(A_4^{6}) \approx -0.874132. \tag{21}$$

The rooted portion of the twenty-four-element Niemeier landscape therefore has the structure

$$23 \text{ rooted types} = \underbrace{A_1^{24}}_{\text{locally stable}} + \underbrace{22 \text{ other rooted types}}_{\text{computationally unstable}}. \tag{22}$$

Remark 6.3. The statement that all twenty-two rooted shells other than A_1^{24} are collectively unstable requires either an explicit Hessian computation for every rooted type or a general theorem implying the result. The present manuscript treats the complete twenty-two-shell instability as a computational claim whose proof is supplied by the accompanying spectral certificates [15], rather than as an analytical deduction from Eq. (18) alone.

6.3 The A_1^{24} shell

The A_1^{24} root system has 48 roots arranged in 24 antipodal pairs, with all distinct non-antipodal roots mutually orthogonal.

For the normalization $R^2 = 2$ ($h = 2$),

$$\lambda_S(2) = \frac{2155 - 832}{3456} = \frac{49}{128}.$$

The tangent dimension is

$$48 \cdot 23 = 1104.$$

The exact collective spectrum is

$$\mathrm{Spec}(H) = \left\{ 0^{(276)}, \left(\frac{17}{64}\right)^{(528)}, \left(\frac{3}{4}\right)^{(276)}, \left(\frac{201}{64}\right)^{(24)} \right\}. \tag{23}$$

Thus

$$H \succeq 0.$$

The 276 zero modes correspond precisely to

$$\dim \mathfrak{so}(24) = \binom{24}{2} = 276,$$

the infinitesimal rotational symmetry of the configuration on S^{23}. The trace is

$$\mathrm{tr}(H) = \frac{3381}{8} = 1104 \cdot \frac{49}{128}.$$

The smallest nonzero eigenvalue is $\frac{17}{64}$, yielding the dimensionless screening ratio

$$\kappa = \frac{17/64}{49/128} = \frac{34}{49}.$$

6.4 The Leech minimal shell

The Leech lattice has 196560 minimal vectors of squared norm 4 after the conventional normalization in which the Leech lattice has minimum norm 4. Thus its minimal vectors lie on S_2^{23}. Equivalently, under the alternative normalization in which the minimum norm is 32, the minimal shell lies on $S^{23}_{\sqrt{32}}$.

In the normalization used for the accompanying spectral computation, the minimal-shell configuration has

$$N = 196560$$

and hence tangent dimension

$$196560 \cdot 23 = 4,520,880.$$

For the chosen normalization, the exact single-particle stiffness reported by the accompanying computation [15] is

$$\lambda_S = \frac{1200199}{196608000} > 0. \tag{24}$$

The corresponding lowest nonzero Hessian eigenvalue is

$$\lambda_{\text{ground}} = \frac{73073}{58982400} > 0. \tag{25}$$

Consequently,

$$\kappa = \frac{\lambda_{\text{ground}}}{\lambda_S} = \frac{730}{3597}.$$

The exact Hessian trace is

$$\mathrm{tr}(H) = \frac{22608148563}{819200}.$$

The complete Hessian decomposes into twelve rational eigenspaces whose multiplicities are governed by the commutant of the relevant Co_0 representation [15].

6.5 Summary of selected shell calculations

7 Global Energy Considerations

Local spherical-shell stability and global lattice optimality are logically distinct notions.

1. **Local spherical stability** concerns the Riemannian Hessian of a finite point configuration on S^{23} under a specified pair potential.

Configuration	N	$\dim \mathcal{T}$	$\min u_{ij}$	λ_S	$\lambda_{\mathrm{ground}}$	**Status**
Rooted, excluding A_1^{24} (22 types)	$24h$	$552h$	2	Eq. (18)	< 0 in certified cases	Unstable
A_1^{24}	48	1104	4	$\frac{49}{128}$	$\frac{17}{64}$	$H \succeq 0$
Λ_{24}	196560	4520880	4	$\frac{1200199}{196608000}$	$\frac{73073}{58982400}$	$H \succeq 0$

Table 1: Selected spectral data for the minimal-shell Hessian under $f(u) = u^{-2}$. All quantities are given in the normalization specified in the surrounding text. The first row summarizes the computationally certified instability of the twenty-two rooted types other than A_1^{24}.

2. **Global lattice optimality** concerns the energy of an infinite periodic point configuration in $\mathbb{R}^{24}$ under completely monotonic functions of squared distance, with the density normalization fixed.

The local potential

$$f(u) = u^{-2}$$

is used only for the finite-shell Hessian problem. If interpreted as an infinite-lattice energy, it corresponds to an inverse fourth-power law

$$\|x\|^{-4},$$

which is not absolutely summable in dimension 24. We therefore do not use $f(u) = u^{-2}$ as the unregularized global lattice energy.

Instead, for the global comparison we may choose

$$f_s(u) = u^{-s}, \qquad s > 12. \tag{26}$$

Then

$$f_s(\|x\|^2) = \|x\|^{-2s},$$

and the condition

$$2s > 24$$

ensures absolute convergence of the corresponding 24-dimensional lattice sum away from the origin.

The function

$$f_s(u) = u^{-s}$$

is completely monotonic on $(0, \infty)$ for every $s > 0$, since

$$(-1)^k f_s^{(k)}(u) = (s)_k \, u^{-s-k} > 0,$$

where $(s)_k$ is the rising factorial.

The universal-optimality theorem of Cohn, Kumar, Miller, Radchenko, and Viazovska establishes the Leech lattice as universally optimal in dimension 24 among periodic configurations of fixed density for completely monotonic potentials of squared distance, under the hypotheses of the theorem [5].

Proposition 7.1 (Global comparison under a convergent potential)**.** *Let*

$$f_s(u) = u^{-s}, \qquad s > 12.$$

Under the density normalization and admissibility hypotheses of the universal-optimality theorem [5], the Leech lattice minimizes the corresponding 24*-dimensional periodic lattice energy. In particular, among the twenty-four Niemeier lattices, no rooted Niemeier lattice has lower energy than the Leech lattice.*

Proof. The function f_s is completely monotonic on $(0, \infty)$ and the condition $s > 12$ ensures absolute convergence of the associated 24-dimensional inverse-power lattice energy.

Universal optimality therefore applies to the Leech lattice under the stated hypotheses. Since every rooted Niemeier lattice is a periodic configuration distinct from the Leech lattice, the Leech lattice attains the universal minimum within this finite Niemeier subset. □

Remark 7.2. The global proposition is deliberately separated from the local Hessian calculation. The purpose is not to claim that the particular shell potential u^{-2} is a fundamental interaction of bosonic string theory, but rather to combine two mathematically well-defined variational questions: local shell stability for one explicit finite-dimensional model, and global lattice optimality for a convergent completely monotonic potential.

7.1 Conditional selection hierarchy

Combining the shell-stability filtration with the global universal-optimality theorem yields the following conditional two-step selection:

$$\mathcal{N}_{24} \xrightarrow{\text{model-dependent shell Hessian}} \{A_1^{24},\ \Lambda_{24}\} \xrightarrow{\text{global universal optimality}} \Lambda_{24}. \tag{27}$$

The first arrow is a computational statement about the particular finite-shell energy functional

$$f_{\text{local}}(u) = u^{-2}.$$

It is not asserted to follow from string-theoretic dynamics.

For the second arrow, we use a completely monotonic potential for which the 24-dimensional lattice energy is absolutely convergent. In particular, one may take

$$f_{\text{global}}(u) = u^{-s}, \qquad s > 12, \tag{28}$$

so that, in terms of ordinary distance r,

$$f_{\text{global}}(r^2) = r^{-2s}, \qquad 2s > 24.$$

The universal-optimality theorem then applies in its convergent lattice-energy regime.

Once Λ_{24} is selected, rootlessness is an exact downstream consequence:

$$K \cong \Lambda_{24} \quad \Longrightarrow \quad \Phi(K) = \varnothing \quad \Longrightarrow \quad \rho_K = 0. \tag{29}$$

Thus the proposed hierarchy should be understood as a conditional mathematical model:

$$\boxed{\begin{array}{c} \text{criticality} \\ +\ \text{lattice assumption} \\ +\ \text{specified variational criteria} \end{array} \quad \Longrightarrow \quad \Lambda_{24}}$$

rather than as a derivation of the Leech lattice from BRST nilpotency alone.

8 The Cannonball Identity as a Lorentzian Coordinate Relation

8.1 The specific Conway coordinate realization

For the Leech lattice, the root system is empty and hence the Weyl-vector contribution vanishes:

$$\rho_\Lambda = 0. \tag{30}$$

In the Lorentzian hyperbolic decomposition

$$\Lambda_{24} \oplus \mathrm{II}_{1,1},$$

the Leech cusp is represented by a primitive null direction. The passage from this abstract hyperbolic decomposition to a diagonal realization of $\mathrm{II}_{25,1}$ is an additional isometric choice.

In the specific diagonal realization used in Conway's Lorentzian construction, the Lorentzian lattice is embedded in $\mathbb{R}^{25,1}$ with diagonal metric

$$(+1^{25} \mid -1),$$

and the relevant primitive null vector is represented by

$$w = (0, 1, 2, \ldots, 24 \mid m). \tag{31}$$

The Lorentzian quadratic form in this realization is

$$w^2 = 0^2 + 1^2 + 2^2 + \cdots + 24^2 - m^2.$$

Since w is null,

$$m^2 = \sum_{k=1}^{24} k^2. \tag{32}$$

The elementary sum-of-squares formula gives

$$\sum_{k=1}^{24} k^2 = \frac{24 \cdot 25 \cdot 49}{6} = 4900. \tag{33}$$

Therefore

$$m^2 = 4900.$$

With the standard positive choice of timelike orientation,

$$m = 70.$$

Hence the Conway coordinate realization yields

$$\boxed{1^2 + 2^2 + \cdots + 24^2 = 70^2.} \tag{34}$$

8.2 Logical status of the identity

It is essential to make the logical dependency explicit:

$$\boxed{\begin{array}{c}\text{Leech cusp} \\ (\rho_\Lambda = 0)\end{array} \quad + \quad \begin{array}{c}\text{specific Conway} \\ \text{coordinate realization}\end{array} \quad + \quad \begin{array}{c}\text{Lorentzian} \\ \text{nullity}\end{array} \quad \Longrightarrow \quad 1^2 + \cdots + 24^2 = 70^2.} \tag{35}$$

The integer 70 is therefore not derived from BRST nilpotency alone. Rather, the critical bosonic string supplies the dimension 24; an additional lattice assumption places the model in the Niemeier landscape; the proposed variational criteria provide a model-dependent route to Λ_{24}; and Conway's specific diagonal realization then identifies the corresponding null vector with

$$(0, 1, 2, \ldots, 24 \mid 70).$$

The last step is geometric rather than dynamical: once the coordinate representative is fixed, the integer 70 follows immediately from Lorentzian nullity.

9 Discussion

The analysis separates several notions that are frequently conflated.

First, critical bosonic string theory fixes the spacetime dimension

$$D = 26,$$

and consequently the light-cone transverse dimension

$$d_\perp = 24.$$

This dimensional statement does not by itself specify a Euclidean lattice.

Second, if one additionally models the relevant 24-dimensional compact sector by an even unimodular positive-definite lattice, the Niemeier classification supplies the finite landscape

$$\mathcal{N}_{24} = \{23 \text{ rooted Niemeier lattices}\} \cup \{\Lambda_{24}\}.$$

Third, the Leech lattice is distinguished algebraically by rootlessness:

$$\Phi(\Lambda_{24}) = \varnothing, \qquad \rho_\Lambda = 0.$$

This is an exact classification statement, not a consequence of BRST criticality.

Fourth, Lorentzian nullity is universal across the Niemeier cusps. For a rooted Niemeier root system with common Coxeter number h,

$$w_R = \rho_R + (h+1)e + hf$$

satisfies

$$w_R^2 = 0.$$

Thus the null cone organizes the corresponding Lorentzian cusp structure, but nullity alone does not distinguish the Leech cusp from the rooted cusps.

Fifth, the finite-shell variational problem under

$$f_{\text{local}}(u) = u^{-2}$$

provides a model-dependent stability filtration. The A_1^{24} shell has an exactly nonnegative collective Hessian, with its zero modes accounted for by infinitesimal rotations. The remaining twenty-two rooted types are claimed to possess negative Hessian directions on the basis of the accompanying exact or certified spectral computations.

Sixth, local shell stability and global lattice optimality are distinct questions. The local calculation concerns a finite configuration on S^{23} and a particular pair potential. The global theorem concerns periodic point configurations in $\mathbb{R}^{24}$ under completely monotonic functions of squared distance. To remain in the ordinary absolutely convergent lattice-energy regime, the global inverse-power example is taken with

$$f_{\text{global}}(u) = u^{-s}, \qquad s > 12.$$

Seventh, once the Leech lattice is selected, Conway's Lorentzian construction provides a distinguished null vector. In the particular diagonal coordinates used here, its nullity becomes

$$0^2 + 1^2 + \cdots + 24^2 - 70^2 = 0,$$

which is precisely the cannonball identity.

The resulting structure is therefore best viewed as a conditional architecture rather than a claim of unique physical derivation:

$$\boxed{\begin{array}{c} \text{BRST criticality} \\ \Downarrow \\ D = 26, \quad d_\perp = 24 \\ \Downarrow \\ \text{additional even-unimodular lattice assumption} \\ \Downarrow \\ \mathcal{N}_{24} \\ \Downarrow \\ \text{model-dependent variational filtration} \\ \Downarrow \\ \{A_1^{24}, \Lambda_{24}\} \\ \Downarrow \\ \text{global universal optimality} \\ \Downarrow \\ \Lambda_{24} \\ \Downarrow \\ \rho_\Lambda = 0 \\ \Downarrow \\ \text{Conway diagonal null vector} \\ \Downarrow \\ 1^2 + \cdots + 24^2 = 70^2. \end{array}} \tag{36}$$

This formulation makes explicit which steps are classification results, which are geometric identities, which are theorem-level global optimization statements, and which depend on the particular variational model adopted in this work.

10 Conclusion

The mathematical structure developed here consists of several connected but logically distinct layers.

Bosonic BRST criticality gives

$$D = 26, \qquad d_\perp = 24.$$

If one additionally assumes an even unimodular positive-definite lattice description of the relevant 24-dimensional sector, the Niemeier classification gives the finite set

$$\mathcal{N}_{24}.$$

Within this landscape, the Leech lattice is uniquely characterized by the absence of roots:

$$\Phi(\Lambda_{24}) = \varnothing, \qquad \rho_\Lambda = 0.$$

The proposed finite-shell variational calculation supplies a model-dependent filtration in which

$$\mathcal{N}_{24} \longrightarrow \{A_1^{24}, \Lambda_{24}\},$$

with the 22 rooted types other than A_1^{24} exhibiting negative Hessian directions according to the accompanying spectral certificates.

A separate global argument based on universal optimality then selects the Leech lattice among the remaining candidates, provided the energy is chosen within the hypotheses and convergence regime of the theorem.

Finally, Conway's Lorentzian construction places the Leech lattice within $\mathrm{II}_{25,1}$. In the particular diagonal coordinate realization used here, the corresponding primitive null vector is

$$w = (0, 1, 2, \dots, 24 \mid 70).$$

Its nullity is exactly

$$w^2 = \sum_{k=1}^{24} k^2 - 70^2 = 0,$$

and therefore

$$\boxed{1^2 + 2^2 + \cdots + 24^2 = 70^2.}$$

The central conclusion is thus conditional but precise:

$$\boxed{\begin{array}{lcl} \text{BRST criticality} & \Longrightarrow & D = 26, \quad d_\perp = 24 \\ & \xRightarrow[\text{additional assumption}]{\text{even unimodular lattice}} & \mathcal{N}_{24} \\ & \xrightarrow{\text{model-dependent shell stability}} & \{A_1^{24}, \Lambda_{24}\} \\ & \xrightarrow{\text{global universal optimality}} & \Lambda_{24} \\ & \Longrightarrow & \rho_\Lambda = 0 \\ & \xrightarrow{\text{Conway diagonal realization}} & w = (0, 1, 2, \dots, 24 \mid 70) \\ & \xrightarrow{w^2=0} & 1^2 + 2^2 + \cdots + 24^2 = 70^2. \end{array}} \tag{37}$$

The integer 24 is therefore supplied by the critical-dimensional setting, while the integer 70 enters only after the additional Lorentzian coordinate realization has been chosen. No claim is made that the cannonball identity, the Leech lattice, or the integer 70 follows from BRST nilpotency alone.

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