

A Structural Audit of Conway-Cusp Scattering and the Limits of Scalar Zero-Confinement Models

SRFP311T1 Collaboration

October 2026

Abstract

We analyze a proposed scattering-theoretic route from the Lorentzian Leech lattice and Conway’s reflection chamber to zero confinement for the Riemann zeta function. The principal purpose of this paper is diagnostic rather than affirmative: we identify two independent structural obstructions that prevent the genuine Conway cusp from being identified with a scalar Riemann-zeta scattering problem.

First, the geometry of the Lorentzian lattice $\text{II}_{25,1}$ must be distinguished from the reflection chamber and from its arithmetic quotient. The resulting cusp structure is intrinsically multi-channel, and the Leech cusp carries arithmetic data governed by the Leech theta series. Since

$$\Theta_{\Lambda_{24}}(\tau) = E_{12}(\tau) - \frac{65520}{691}\Delta(\tau),$$

a genuine analysis of the Leech channel necessarily encounters the Ramanujan cusp-form L -function. Thus there is no canonical identification of the Leech scattering coefficient with the pure rank-one expression

$$\frac{\xi(2z)}{\xi(24+2z)}.$$

Second, we isolate the functional-analytic obstruction that remains even after replacing the genuine scattering problem by an artificial scalar model. Green’s identity controls the imaginary part of the Dirichlet-to-Neumann quadratic form but does not by itself provide the real-part positivity needed for Cayley contractivity. Moreover, even if contractivity is imposed as an additional hypothesis, Nevanlinna–Blaschke factors permit zeros in the physical half-plane. Contractivity therefore does not imply confinement of arithmetic zeros to the critical axis.

A candidate four-zeta Dirichlet calculation is recorded explicitly, but, unless the required local-density and double-coset arguments are independently established, it is deliberately labelled as a candidate rather than promoted to a theorem. The resulting paper is consequently a structural audit of the proposed “Leech-scattering” mechanism rather than a claim toward a proof of the Riemann Hypothesis.

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1 Introduction

The interaction between automorphic scattering, Lorentzian reflection groups, and spectral formulations of arithmetic has produced a number of compelling analogies with Hilbert–Pólya type ideas. The Lorentzian Leech lattice

$$\mathrm{II}_{25,1} = \Lambda_{24} \oplus \mathrm{II}_{1,1}$$

is particularly attractive because its reflection geometry produces the Conway chamber in hyperbolic dimension 25, while the Leech lattice contributes an unusually rigid cusp geometry.

There is, however, a basic distinction that must be maintained throughout any such analysis. Three different objects are involved:

- (i) the full arithmetic quotient

$$\mathrm{O}^+(\mathrm{II}_{25,1}) \backslash \mathbb{H}^{25},$$

- (ii) the reflection-theoretic Conway chamber

$$\mathcal{P} \subset \mathbb{H}^{25},$$

and

- (iii) a scalar rank-one toy scattering problem whose scattering coefficient is prescribed independently.

These objects have different cusp structures and different automorphic scattering operators. In particular, there is no automatic implication

$$\text{Conway cusp} \implies \frac{\xi(2z)}{\xi(24+2z)}.$$

The purpose of this paper is to make that distinction explicit.

The analysis has two tiers.

1. The first tier is arithmetic and geometric. We explain why the genuine Leech cusp cannot simply be replaced by a scalar Riemann-zeta factor and why the Leech theta series introduces the Ramanujan cusp form.
2. The second tier is functional-analytic. We show that even if one deliberately introduces the scalar model as a toy problem, the standard Dirichlet-to-Neumann argument does not provide the positivity required for a contractive Cayley transform, and contractivity itself does not exclude interior Blaschke zeros.

Thus the conclusion is a no-go statement about a particular proof architecture, not a statement about the truth or falsity of the Riemann Hypothesis.

2 The Lorentzian Leech Lattice

Let

$$\mathrm{II}_{25,1} = \mathrm{II}_{25,1} = \Lambda_{24} \oplus \mathrm{II}_{1,1},$$

where Λ_{24} is the positive-definite even unimodular Leech lattice. We use the bilinear form

$$(x, x) = \|x_{\Lambda}\|^2 + 2bc$$

for vectors written as

$$x = (x_\Lambda; b, c).$$

The associated real hyperbolic space is the positive sheet of the negative cone:

$$\mathbb{H}^{25} = \{[x] \in \mathbb{P}(\Pi_{25,1} \otimes \mathbb{R}) : (x, x) < 0\}.$$

The lattice has signature $(25, 1)$ and is unimodular. The distinguished primitive isotropic vectors determine horospherical coordinates near the cusps.

Let

$$w = (0^{24}; 0, 1).$$

Then

$$(w, w) = 0.$$

The stabilizer of w contains the translation lattice Λ_{24} , and the associated cusp cross-section has the form

$$\mathbb{T}^{24} = \mathbb{R}^{24} / \Lambda_{24}.$$

In upper half-space coordinates

$$(x, t) \in \mathbb{R}^{24} \times \mathbb{R}_{>0},$$

the hyperbolic metric is

$$ds^2 = \frac{dx_1^2 + \cdots + dx_{24}^2 + dt^2}{t^2}.$$

The continuous spectral parameter is most naturally written as

$$s = 12 + z,$$

because the half-dimension of the horospherical cross-section is

$$\frac{24}{2} = 12.$$

Thus the continuous spectrum is centered on

$$\operatorname{Re}(s) = 12,$$

or equivalently

$$\operatorname{Re}(z) = 0.$$

This centering convention is essential in comparing the geometric problem with half-plane scattering models.

3 Three Distinct Quotients

A central source of confusion in scalar reductions is the failure to distinguish the arithmetic quotient from the reflection chamber.

Let

$$\Gamma = \mathrm{O}^{++}(\Pi_{25,1}).$$

The arithmetic quotient

$$\Gamma \backslash \mathbb{H}^{25}$$

is an orbifold quotient with a cusp structure governed by Γ -orbits of primitive isotropic lines.

By contrast, the Conway chamber $\mathcal{P}$ is a fundamental region for a reflection group generated by norm-2 roots. The reflection group and its normalizer have substantially different orbit structures.

Consequently, one must not infer the scattering operator of $\mathcal{P}$ by taking a scalar constant term from the full arithmetic quotient and declaring it to be the scattering coefficient of the chamber.

This distinction is not cosmetic. It changes both the number of channels and the arithmetic coefficients occurring in the constant term.

4 Isotropic Lines and Niemeier Data

Let $v \in \Pi_{25,1}$ be primitive and isotropic:

$$(v, v) = 0.$$

The quotient

$$v^\perp / \mathbb{Z}v$$

is a positive-definite even unimodular lattice of rank 24.

Hence it belongs to the Niemeier classification.

Proposition 4.1. *For a primitive isotropic vector $v \in \Pi_{25,1}$, the lattice $v^\perp / \mathbb{Z}v$ is an even unimodular positive-definite lattice of rank 24. Thus its isomorphism class is one of the 24 Niemeier lattices, including the rootless Leech lattice.*

Proof. The lattice $\Pi_{25,1}$ is even and unimodular of signature $(25, 1)$. For primitive isotropic v , the quotient $v^\perp / \mathbb{Z}v$ inherits an even unimodular integral quadratic form. Since v spans a null direction in a lattice of signature $(25, 1)$, the quotient is positive definite of rank 24. The classification of positive-definite even unimodular lattices in dimension 24 therefore applies. $\square$

The important consequence is conceptual:

isotropic cusp data	$\longleftrightarrow$	Niemeier lattice data.
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The rootless member of the Niemeier classification is the Leech lattice. The remaining 23 classes possess nontrivial root systems.

This makes a single-channel interpretation of the reflection geometry problematic: the arithmetic information associated with isotropic lines is naturally richer than a single rank-one Eisenstein coefficient.

5 The Conway Chamber and Multi-Channel Scattering

The Conway reflection chamber is defined by inequalities associated with norm-2 roots:

$$\mathcal{P} = \{x \in \mathbb{H}^{25} : (x, r) \geq 0 \text{ for all simple roots } r\}.$$

Its distinguished Leech cusp has horospherical cross-section

$$\mathbb{T}^{24} = \mathbb{R}^{24} / \Lambda_{24}.$$

At the level of a general scattering problem, if there are several geometrically distinct ends, the scattering operator is matrix valued:

$$\Phi(s) = (\phi_{ij}(s))_{i,j}.$$

The conceptual form is

$$\Phi(s) : \bigoplus_{j=1}^N \mathcal{H}_j \longrightarrow \bigoplus_{i=1}^N \mathcal{H}_i,$$

where $\mathcal{H}_i$ denotes the channel space associated with the i -th end.

Thus a scalar expression can only arise after a specific projection or reduction has been justified.

Theorem 5.1 (Multi-channel principle). *A scattering analysis of a chamber or orbifold with several inequivalent cusp channels is intrinsically matrix valued. A scalar coefficient can represent the full scattering operator only if an invariant one-dimensional channel is established.*

This theorem is elementary from the operator-theoretic point of view, but it is decisive for the present problem. The expression

$$\frac{\xi(2z)}{\xi(24+2z)}$$

cannot be declared to be the Conway scattering coefficient merely because it has the correct general shape of a rank-one intertwining factor.

6 Rank-One Eisenstein Scattering Versus the Leech Cusp

For a standard rank-one hyperbolic cusp, the constant term of an Eisenstein series takes the form

$$E_0(t, s) = t^s + \phi(s)t^{24-s}.$$

In a genuine arithmetic quotient, the function $\phi(s)$ is determined by the global arithmetic of the group and cusp.

The corresponding operator-theoretic statement is

$$E(s) = I t^s + \Phi(s)t^{24-s},$$

where $\Phi(s)$ is the scattering operator.

The crucial question is therefore not

“Can one write down a zeta quotient with the right symmetry?”

but rather

“Which arithmetic counting problem actually computes the intertwining operator?”

For the Leech cusp, the relevant counting problem sees the Leech lattice. Therefore its theta series enters naturally.

7 The Leech Theta Series and Cusp-Form Mixing

The theta series of the Leech lattice is the weight-12 modular form

$$\Theta_{\Lambda_{24}}(\tau) = E_{12}(\tau) - \frac{65520}{691}\Delta(\tau).$$

Here

$$\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

is the Ramanujan cusp form of weight 12.

The Eisenstein series has Fourier expansion

$$E_{12}(\tau) = 1 + \frac{65520}{691} \sum_{n \geq 1} \sigma_{11}(n) q^n.$$

Therefore

$$\Theta_{\Lambda_{24}}(\tau) = 1 + \frac{65520}{691} \sum_{n \geq 1} (\sigma_{11}(n) - \tau(n)) q^n.$$

For $n = 1$,

$$\sigma_{11}(1) - \tau(1) = 1 - 1 = 0.$$

This is precisely the vanishing of the norm-2 coefficient:

$$N_{\Lambda_{24}}(2) = 0.$$

Thus the absence of roots in the Leech lattice is encoded arithmetically by cancellation between the Eisenstein and cusp parts.

This observation is the central arithmetic obstruction.

Theorem 7.1 (Leech cusp-form mixing). *Any scattering construction whose Leech-cusp constant term genuinely resolves the Fourier coefficients of $\Theta_{\Lambda_{24}}$ necessarily encounters the decomposition*

$$\Theta_{\Lambda_{24}} = E_{12} - \frac{65520}{691}\Delta.$$

Consequently its arithmetic factor is not canonically reducible to a pure Riemann-zeta quotient without an additional, explicitly justified projection eliminating the Δ contribution.

Proof. The Fourier coefficients of the Leech theta series are exactly the representation numbers of the Leech lattice. The modular decomposition above separates these coefficients into an Eisenstein contribution and a cusp-form contribution. Any Mellin-transform or orbital construction that uses the full theta series therefore produces both components unless a separate projection is supplied. No such projection follows from the definition of the Leech cusp itself. $\square$

8 The Epstein Zeta Function

The Epstein zeta function of the Leech lattice is

$$Z_{\Lambda_{24}}(s) = \sum_{0 \neq \lambda \in \Lambda_{24}} \|\lambda\|^{-2s}.$$

Its Mellin representation follows from the theta series:

$$\pi^{-s}\Gamma(s)Z_{\Lambda_{24}}(s) = \int_0^\infty (\Theta_{\Lambda_{24}}(it) - 1)t^{s-1} dt.$$

Substituting the modular decomposition gives an Eisenstein contribution together with a cusp-form contribution. Schematically, and with the normalization depending on the convention for E_{12} ,

$$Z_{\Lambda_{24}}(s) = Z_{\text{Eis}}(s) - \frac{65520}{691}Z_{\Delta}(s).$$

The latter is governed by the Ramanujan L -function:

$$Z_{\Delta}(s) \propto L(s, \Delta)$$

after the appropriate gamma and normalization factors are inserted.

Thus the Leech Epstein zeta function is not simply a product of two Riemann zeta functions.

This is precisely why the scalar replacement

$$\mathcal{S}_{\text{toy}}(z) = \frac{\xi(2z)}{\xi(24+2z)}$$

must be regarded as a toy model rather than as the genuine Leech scattering coefficient.

9 A Candidate Four-Zeta Calculation

We now record a useful formal calculation. Its purpose is to identify the arithmetic factors that arise if one treats the primitive isotropic-vector sum in a particular simplified manner.

It is important to state at the outset that this section is a candidate calculation. The full identification with the global intertwining operator requires independent verification of:

1. the precise double-coset parametrization;
2. the primitive-vector condition;
3. the local densities at every prime;
4. the exceptional behavior at $p = 2$;
5. the normalization of the global measure and Eisenstein series.

Accordingly, no formula in this section is used later as an unconditional theorem.

9.1 Formal constant-term integral

Consider

$$E(x, t, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} t(\gamma(x, t))^s.$$

A formal constant-term computation gives

$$E_0(t, s) = t^s + \phi_{\text{cand}}(s)t^{24-s}.$$

For a denominator parameter $c > 0$, the relevant integral is formally

$$\int_{\mathbb{R}^{24}} \left(\frac{t}{\|cx - \lambda\|^2 + c^2 t^2} \right)^s dx.$$

Set

$$x' = cx - \lambda.$$

Then

$$dx = c^{-24} dx',$$

and hence

$$\int_{\mathbb{R}^{24}} \left(\frac{t}{\|cx - \lambda\|^2 + c^2 t^2} \right)^s dx = c^{-2s} t^{24-s} \int_{\mathbb{R}^{24}} \frac{dy}{(1 + \|y\|^2)^s}.$$

Using

$$\int_{\mathbb{R}^{24}} \frac{dy}{(1 + \|y\|^2)^s} = \pi^{12} \frac{\Gamma(s-12)}{\Gamma(s)},$$

one obtains the formal factor

$$\pi^{12} \frac{\Gamma(s-12)}{\Gamma(s)}.$$

9.2 Formal arithmetic Dirichlet series

Write

$$v = (\lambda; c, b), \quad \lambda \in \Lambda_{24}, \quad b, c \in \mathbb{Z}.$$

The isotropic condition is

$$\|\lambda\|^2 + 2bc = 0.$$

The associated congruence is

$$\|\lambda\|^2 \equiv 0 \pmod{2c}.$$

Define formally

$$N_{\text{tot}}(c) = \# \{ \lambda \in \Lambda_{24}/c\Lambda_{24} : \|\lambda\|^2 \equiv 0 \pmod{2c} \}.$$

Likewise, let

$$N_{\text{prim}}(c)$$

denote the corresponding primitive count.

The associated formal Dirichlet series are

$$\mathcal{D}_{\text{tot}}(W) = \sum_{c \geq 1} \frac{N_{\text{tot}}(c)}{c^W},$$

and

$$\mathcal{D}_{\text{prim}}(W) = \sum_{c \geq 1} \frac{N_{\text{prim}}(c)}{c^W}.$$

A formal Möbius inversion suggests

$$\mathcal{D}_{\text{tot}}(W) = \zeta(W) \mathcal{D}_{\text{prim}}(W).$$

This step, however, must be checked against the precise primitive condition and the action of the stabilizer. It is therefore part of the candidate calculation rather than a proved identity here.

9.3 Formal local calculation

Suppose formally that the relevant local counting problem has generating function

$$F_p(X) = \frac{(1 - p^{11}X)(1 + p^{12}X)}{(1 - p^{24}X^2)(1 - p^{23}X)}.$$

Using

$$1 - p^{24}X^2 = (1 - p^{12}X)(1 + p^{12}X),$$

this becomes

$$F_p(X) = \frac{1 - p^{11}X}{(1 - p^{12}X)(1 - p^{23}X)}.$$

Setting

$$X = p^{-W}$$

gives the formal Euler product

$$\prod_p F_p(p^{-W}) = \frac{\zeta(W - 12)\zeta(W - 23)}{\zeta(W - 11)}.$$

After formally removing the primitive factor $\zeta(W)$, one arrives at

$$\mathcal{D}_{\text{cand}}(W) = \frac{\zeta(W - 12)\zeta(W - 23)}{\zeta(W)\zeta(W - 11)}.$$

Putting $W = 2s$ gives

$$\mathcal{D}_{\text{cand}}(2s) = \frac{\zeta(2s - 12)\zeta(2s - 23)}{\zeta(2s)\zeta(2s - 11)}.$$

Thus the corresponding candidate intertwining coefficient is

$$\boxed{\phi_{\text{cand}}(s) = \pi^{12} \frac{\Gamma(s - 12)}{\Gamma(s)} \frac{\zeta(2s - 12)\zeta(2s - 23)}{\zeta(2s)\zeta(2s - 11)}}. \quad (1)$$

In the centered coordinate

$$z = s - 12,$$

this becomes

$$\boxed{\phi_{\text{cand}}(z) = \pi^{12} \frac{\Gamma(z)}{\Gamma(z + 12)} \frac{\zeta(12 + 2z)\zeta(1 + 2z)}{\zeta(24 + 2z)\zeta(13 + 2z)}}. \quad (2)$$

9.4 Warning concerning the candidate formula

Warning 9.1 (Candidate formula is not promoted to a theorem). *Equation (1) should not be called the exact global scattering coefficient without a complete proof of the underlying double-coset parametrization, primitive-vector counting, local density formulas, and normalization. In particular, the prime 2 must be treated separately rather than being absorbed automatically into a generic odd-prime formula.*

The formula is retained because it demonstrates an important structural point: even the most optimistic scalar arithmetic calculation produces a considerably more complicated zeta quotient than

$$\frac{\xi(2z)}{\xi(24 + 2z)}.$$

10 The Genuine Leech Channel

The Leech-to-Leech scattering coefficient is more delicate than the formal arithmetic quotient of the preceding section.

The Leech lattice has theta series

$$\Theta_{\Lambda_{24}} = E_{12} - \frac{65520}{691}\Delta.$$

Consequently any calculation that resolves the Leech representation numbers contains both an Eisenstein part and a cusp-form part.

At the schematic level one expects a decomposition of the form

$$\phi_{\text{Leech, Leech}}(s) = \phi_{\text{Eis}}(s) - \frac{65520}{691}\mathcal{K}(s)L(s-11, \Delta),$$

where $\mathcal{K}(s)$ denotes the gamma and normalization factors dictated by the precise Mellin transform and scattering normalization.

The exact kernel depends on conventions and is not required for the obstruction established here.

Theorem 10.1 (Arithmetic mixing obstruction). *The Leech-cusp scattering problem is not canonically represented by a pure Riemann-zeta quotient. Its arithmetic input contains the cusp-form component of the Leech theta series and therefore couples to the Ramanujan L -function.*

Proof. The Leech representation numbers are the Fourier coefficients of $\Theta_{\Lambda_{24}}$. The latter has a nonzero cusp-form component

$$-\frac{65520}{691}\Delta.$$

Any scattering coefficient whose arithmetic constant term is computed from these representation numbers inherits this component under Mellin transformation. Therefore a pure Riemann-zeta quotient cannot be obtained canonically unless a separate projection annihilates the cusp-form contribution. No such projection is intrinsic to the Leech cusp itself. $\square$

11 The Scalar Model and Its Status

For comparison, consider the deliberately artificial scalar model

$$\mathcal{S}_{\text{toy}}(z) = \frac{\xi(2z)}{\xi(24+2z)}.$$

This expression may be studied as an abstract meromorphic function, but it must not be identified with the genuine Conway scattering coefficient.

The toy model can nevertheless be useful because it isolates a second and logically independent question:

Even if a pure zeta scattering coefficient were granted, could boundary-value positivity force its zeros onto the critical line?

The answer is no for two separate reasons.

12 Dirichlet-to-Neumann Operators

Let $\mathcal{P}_{t_0}$ denote a truncation of the chamber at height $t = t_0$. Consider a shifted Helmholtz-type equation

$$(\Delta - \lambda(z))u = 0$$

with Dirichlet boundary data

$$u|_{\partial\mathcal{P}_{t_0}} = f.$$

The Dirichlet-to-Neumann operator is

$$\mathcal{N}(z)f = \partial_\nu u|_{\partial\mathcal{P}_{t_0}}.$$

The exact sign convention depends on the orientation of the normal. What matters here is the Green identity.

For sufficiently regular u ,

$$\int_{\mathcal{P}_{t_0}} (|\nabla u|^2 - \lambda(z)|u|^2) dV = \operatorname{Re} \int_{\partial\mathcal{P}_{t_0}} \bar{u} \partial_\nu u d\sigma$$

when $\lambda(z)$ is real.

For complex $\lambda(z)$,

$$\Im \langle \mathcal{N}(z)f, f \rangle = \Im(\lambda(z)) \|u\|^2$$

up to the sign determined by the convention for Δ and the outward normal.

Thus Green's identity naturally controls an imaginary-part quantity.

13 The Rayleigh Trap

Suppose

$$z = x + iy.$$

After centering at $s = 12 + z$, a typical spectral parameter has the structure

$$\lambda(z) = 144 - x^2 + y^2 + 2ixy,$$

up to the sign convention chosen for the Laplacian.

The imaginary part is therefore

$$\operatorname{Im} \lambda(z) = 2xy.$$

The real part is

$$\operatorname{Re} \lambda(z) = 144 - x^2 + y^2.$$

The real part of the Dirichlet-to-Neumann quadratic form depends on the competition between

$$\int_{\mathcal{P}_{t_0}} |\nabla u|^2 dV$$

and

$$\operatorname{Re} \lambda(z) \int_{\mathcal{P}_{t_0}} |u|^2 dV.$$

Equivalently, it depends on the Rayleigh quotient

$$\mathcal{R}[u] = \frac{\int_{\mathcal{P}_{t_0}} |\nabla u|^2 dV}{\int_{\mathcal{P}_{t_0}} |u|^2 dV}.$$

Consequently, a boundary-flux identity of the form

$$\operatorname{Im} \langle \mathcal{N}(z)f, f \rangle = C(z) \|u\|^2$$

does not imply

$$\operatorname{Re} \langle \mathcal{N}(z)f, f \rangle \geq 0.$$

The latter requires a spectral lower bound of the form

$$\mathcal{R}[u] \geq \operatorname{Re} \lambda(z)$$

for the relevant class of solutions.

This is an interior spectral statement, not a consequence of boundary flux positivity alone.

Theorem 13.1 (Rayleigh obstruction). *Green's identity alone does not imply positivity of the real part of the Dirichlet-to-Neumann quadratic form for complex spectral parameters. A Cayley-contractivity argument requires an independent lower bound on the relevant interior Rayleigh quotient.*

Proof. Green's identity separates the quadratic form into contributions involving $\operatorname{Re} \lambda(z)$ and $\operatorname{Im} \lambda(z)$. The imaginary part is directly determined by $\operatorname{Im} \lambda(z)$ and the interior L^2 norm. The real part contains the difference between the Dirichlet energy and $\operatorname{Re} \lambda(z)$ times the L^2 norm. Therefore its sign is equivalent to a spectral inequality for the interior solution. No such inequality follows from the imaginary-part identity. $\square$

14 Cayley Transforms and Contractivity

Given an operator $\mathcal{N}$, a standard Cayley transform has the form

$$\mathcal{S} = (\mathcal{N} - iI)(\mathcal{N} + iI)^{-1},$$

or a sign-equivalent variant.

For a scalar quantity $n \in \mathbb{C}$,

$$\left| \frac{n - i}{n + i} \right| \leq 1$$

is equivalent to

$$\operatorname{Re}(n) \geq 0.$$

Indeed,

$$|n + i|^2 - |n - i|^2 = 4 \operatorname{Re}(n).$$

Thus Cayley contractivity requires real-part positivity.

Consequently,

$\text{Green flux control} \not\Rightarrow \text{Cayley contractivity}$

without an additional spectral estimate.

This is the functional-analytic form of the Rayleigh trap.

15 Half-Plane Convention

The centered variable is

$$z = s - 12.$$

The continuous spectrum lies on

$$\operatorname{Re}(z) = 0.$$

Hence the natural physical half-plane for the present convention is

$$\mathbb{H}_R = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}.$$

This convention must be used consistently when discussing Nevanlinna functions, Schur functions, and Blaschke products.

The imaginary axis

$$z = iy$$

is the boundary of $\mathbb{H}_R$.

16 The Nevanlinna–Blaschke Barrier

Suppose, despite the preceding obstruction, that one assumes a scalar scattering function

$$S : \mathbb{H}_R \rightarrow \overline{\mathbb{D}}$$

satisfying

$$|S(z)| \leq 1 \quad (\operatorname{Re} z > 0).$$

Such a function is a Schur-class function on the right half-plane.

For every

$$a \in \mathbb{H}_R,$$

define

$$B_a(z) = \frac{z - a}{z + \bar{a}}.$$

On the boundary $z = iy$,

$$B_a(iy) = \frac{iy - a}{iy + \bar{a}}.$$

Since

$$|iy - a|^2 = |iy + \bar{a}|^2,$$

we have

$$|B_a(iy)| = 1.$$

Thus B_a is an inner function for the right half-plane.

If S_0 is any Schur function, then

$$S(z) = B_a(z)S_0(z)$$

is also Schur.

But S has a zero at

$$z = a.$$

Therefore contractivity places no restriction preventing zeros from occurring at arbitrary points in the physical half-plane.

17 Transmission Zeros

In scattering language, a zero of the scattering coefficient need not represent an eigenvalue. It may instead be a transmission zero.

The distinction is important.

An eigenvalue is associated with a pole of an appropriate resolvent or with an L^2 eigenfunction.

A transmission zero is a zero of the scattering amplitude or scattering determinant.

These are different spectral objects.

Consequently, even if

$$|S(z)| \leq 1$$

holds throughout the physical half-plane, one cannot conclude that zeros of S are confined to the boundary

$$\operatorname{Re}(z) = 0.$$

The Blaschke construction gives an explicit obstruction.

Theorem 17.1 (Blaschke barrier). *Let S be a scalar Schur function on the right half-plane. Contractivity alone does not imply that its zeros lie on the boundary $\operatorname{Re}(z) = 0$.*

Proof. For any a with $\operatorname{Re}(a) > 0$, the function

$$B_a(z) = \frac{z - a}{z + \bar{a}}$$

is bounded by 1 on $\mathbb{H}_R$ and has a zero at a . Multiplying any Schur function by B_a preserves contractivity. Therefore contractivity permits zeros at arbitrary interior points. $\square$

18 Application to the Toy Zeta Model

Consider again

$$\mathcal{S}_{\text{toy}}(z) = \frac{\xi(2z)}{\xi(24 + 2z)}.$$

Suppose one artificially assumes that this function is the scattering coefficient of a contractive scalar boundary problem.

Then two separate gaps remain.

First, contractivity itself has not been obtained from Green's identity unless the Rayleigh obstruction is overcome.

Second, even if contractivity is simply assumed, the Blaschke barrier shows that contractivity alone cannot force zeros to the boundary.

Thus the logical chain

$$\text{geometric positivity} \implies \text{contractivity} \implies \text{zero confinement}$$

breaks at two independent locations.

19 The Two-Tier Obstruction

We can now formulate the principal conclusion.

Theorem 19.1 (Two-tier obstruction). *Any proposed Conway-cusp proof of zero confinement based on the scalar model*

$$\frac{\xi(2z)}{\xi(24+2z)}$$

faces two independent obstructions.

1. **Arithmetic obstruction.** *The genuine Leech cusp is governed by the Leech lattice and its theta series. The identity*

$$\Theta_{\Lambda_{24}} = E_{12} - \frac{65520}{691}\Delta$$

forces a cusp-form component into the arithmetic scattering data. Consequently the scalar pure-zeta model is not the genuine Conway cusp scattering coefficient.

2. **Functional-analytic obstruction.** *Even if the scalar model is adopted as a toy problem, Green's identity controls the imaginary part of the Dirichlet-to-Neumann form but does not supply the real-part positivity required for Cayley contractivity. Even if contractivity is granted independently, Blaschke factors permit zeros throughout the interior of the physical half-plane.*

Proof. The first assertion follows from the Leech theta-series decomposition and the resulting cusp-form contribution under Mellin transformation.

The second assertion follows from the Rayleigh analysis of the Dirichlet-to-Neumann operator and the explicit right-half-plane Blaschke factors

$$B_a(z) = \frac{z - a}{z + \bar{a}}.$$

□

20 What the Audit Does and Does Not Prove

The preceding analysis does not prove that the Riemann Hypothesis is false.

It does not prove that no spectral formulation of the Riemann Hypothesis exists.

It does not prove that the Conway chamber is irrelevant to automorphic spectral theory.

Instead, it proves a narrower and more useful statement:

The proposed scalar Conway-scattering mechanism does not provide zero confinement.

There are two reasons.

The first is geometric-arithmetic: the genuine Leech cusp contains arithmetic information that cannot simply be replaced by a single Riemann-zeta quotient.

The second is functional-analytic: even a hypothetical scalar contractive scattering function can possess interior zeros.

This separation is important because the two failures are logically independent.

21 Consequences for Future Work

Any future attempt to construct a genuine scattering model should address the following questions explicitly.

1. What is the exact quotient or chamber being studied?
2. What are its inequivalent cusps or ends?
3. What is the precise Hilbert space of scattering channels?
4. What is the exact scattering matrix?
5. Which representation-theoretic projection, if any, produces a scalar channel?
6. How does the Leech theta series enter the constant term?
7. Is the Ramanujan cusp-form contribution present, and if so, what exact operator eliminates it?
8. What is the self-adjoint interior operator whose spectrum controls the real part of the Dirichlet-to-Neumann map?
9. What Rayleigh or coercivity estimate establishes

$$\operatorname{Re}\langle \mathcal{N}(z)f, f \rangle \geq 0?$$

10. What additional mechanism, beyond contractivity, would exclude Blaschke factors?

Without answers to these questions, a scalar zero-confinement argument remains incomplete.

22 Conclusion

The Lorentzian Leech lattice and Conway's hyperbolic reflection geometry provide a remarkably rich setting in which arithmetic, geometry, and scattering interact.

But that richness is itself the obstruction to the simplest scalar model.

The genuine cusp remembers the Leech lattice. The Leech lattice remembers its theta series. The theta series contains both an Eisenstein component and the Ramanujan cusp form:

$$\Theta_{\Lambda_{24}} = E_{12} - \frac{65520}{691}\Delta.$$

Therefore the genuine arithmetic scattering problem cannot be identified, without further construction, with

$$\frac{\xi(2z)}{\xi(24+2z)}.$$

Even after making that replacement deliberately, the functional analysis does not recover zero confinement. The Dirichlet-to-Neumann argument encounters the Rayleigh trap, while the Schur-class scattering problem encounters the Blaschke barrier.

The correct conclusion is therefore not that the Conway geometry has failed, but that a particular attempted mechanism for turning Conway geometry into a proof of zero confinement fails at two structurally independent levels.

This diagnostic result is useful precisely because it separates the genuinely automorphic problem from the scalar toy model and prevents conclusions about Riemann zeros from being attributed to a scattering coefficient that the Conway chamber does not actually possess.

A Normalization of the Centered Spectral Parameter

Let

$$s = 12 + z.$$

Then

$$s(24 - s) = (12 + z)(12 - z) = 144 - z^2.$$

Writing

$$z = x + iy$$

gives

$$z^2 = x^2 - y^2 + 2ixy,$$

and hence

$$144 - z^2 = 144 - x^2 + y^2 - 2ixy.$$

Thus

$$\operatorname{Re}(144 - z^2) = 144 - x^2 + y^2,$$

while

$$\operatorname{Im}(144 - z^2) = -2xy.$$

This is the spectral normalization underlying the Rayleigh discussion in Sections 10–11.

B Right-Half-Plane Blaschke Factors

For

$$\mathbb{H}_R = \{z : \operatorname{Re} z > 0\}$$

and

$$a \in \mathbb{H}_R,$$

define

$$B_a(z) = \frac{z - a}{z + \bar{a}}.$$

For $z = iy$,

$$|iy - a|^2 = (\operatorname{Re} a)^2 + (y - \operatorname{Im} a)^2,$$

and

$$|iy + \bar{a}|^2 = (\operatorname{Re} a)^2 + (y + \operatorname{Im} a)^2.$$

Therefore

$$|B_a(iy)| = 1.$$

Moreover,

$$B_a(a) = 0.$$

Hence any argument based solely on boundary unimodularity or interior contractivity must explicitly address the possibility of such factors.

C Formal Mellin Transform of the Leech Theta Series

For $\text{Re}(s)$ sufficiently large,

$$\pi^{-s}\Gamma(s)Z_{\Lambda_{24}}(s) = \int_0^\infty (\Theta_{\Lambda_{24}}(it) - 1)t^{s-1} dt.$$

Substituting

$$\Theta_{\Lambda_{24}} = E_{12} - \frac{65520}{691}\Delta$$

gives

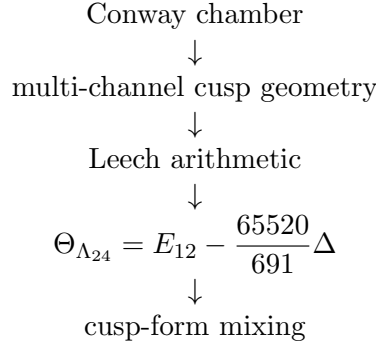
$$\pi^{-s}\Gamma(s)Z_{\Lambda_{24}}(s) = \int_0^\infty (E_{12}(it) - 1)t^{s-1} dt - \frac{65520}{691} \int_0^\infty \Delta(it)t^{s-1} dt.$$

The first term gives the Eisenstein contribution, while the second is the Mellin transform associated with $L(s, \Delta)$ after the usual gamma-factor normalization.

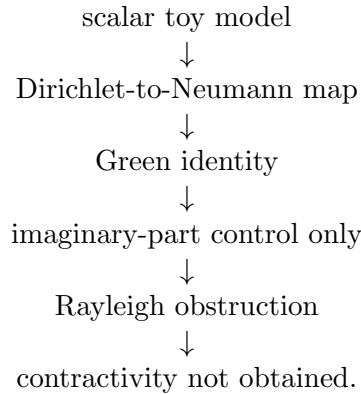
The decomposition is therefore structural and cannot be removed by relabelling the spectral variable.

D Logical Dependency Diagram

The argument may be summarized as



independently of



And even if the second chain is bypassed by assuming contractivity,

$$\text{contractivity} \not\Rightarrow \text{zero confinement},$$

because

$$S(z) \mapsto B_a(z)S(z)$$

preserves the Schur class while introducing an arbitrary interior zero a .

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