

An Explicit D_4 –Leech–MOG–Hexacode Bridge

Root-Shell Reduction, Triality, and Hexacode Complementarity

SRFP311T1 Collaboration

October 2026

Abstract

We give an explicit coordinate realization of a scaled D_4 root lattice inside the Leech lattice Λ_{24} , using the standard Conway–Sloane Construction-A coordinate frame. Four explicit vectors of squared norm 4 are exhibited whose Gram matrix is exactly $2G_{D_4}$. Their 24 root images decompose into three eight-element sectors corresponding to the vector and two half-spinor sectors of D_4 .

We then choose an explicit Miracle Octad Generator (MOG) sextet whose intersection with the relevant Golay octad has shape $(2, 2, 2, 2, 0, 0)$. This choice avoids the rank collapse that occurs when the octad is the union of two complete tetrads. The resulting coordinate description distinguishes the ambient $\mathbb{F}_4^6$ representation from the distinguished Hexacode $\mathcal{H}_6 \subset \mathbb{F}_4^6$.

The reduction on the D_4 root shell is deliberately treated as type-sensitive rather than as a global linear quotient. This makes the triality structure transparent: the three eight-element root sectors have identical metric distributions, while the two spinor sectors collapse to the same weight-four Hexacode line under the ordinary Construction-A projection. We conclude with the exact complementarity relation

$$\text{span}_{\mathbb{F}_4}(\rho(D_4)) + \mathcal{H}_6 = \mathbb{F}_4^6, \quad \text{span}_{\mathbb{F}_4}(\rho(D_4)) \cap \mathcal{H}_6 = \mathbb{F}_4(1, 1, 1, 1, 0, 0).$$

An appendix records the separate numerical elliptic-curve construction only as an audited, non-intrinsic discretization attempt; it is not used in the main bridge.

1 Scope and Mathematical Architecture

The purpose of this note is to construct an explicit bridge between three classical structures:

$$\sqrt{2}D_4 \hookrightarrow \Lambda_{24} \longrightarrow \text{MOG coordinate data} \subset \mathbb{F}_4^6.$$

The construction has three logically distinct layers.

1. **Lattice layer.** An explicit 24×4 matrix V embeds a scaled D_4 root system into the minimal shell of the Leech lattice.
2. **Coding layer.** An explicit Golay octad and MOG sextet identify four distinguished duads with four tetrads of the MOG. The ordinary Construction-A reduction and the shell-level reduction are kept conceptually separate.
3. **Triality layer.** The 24 roots split into three eight-element sectors with identical metric behavior. These sectors realize the 8_v , 8_s , and 8_c triality decomposition of D_4 .

No arithmetic modulus, elliptic curve, complex multiplication condition, or numerical discretization is required for this intrinsic construction.

In particular, the map from the lattice shell to the MOG coordinates is not asserted to be a single linear map on the whole lattice. This distinction is essential and will be made explicit below.

2 The Leech Coordinate Frame

We work in the standard Conway–Sloane Construction-A coordinate frame

$$\Lambda_{24} \subset \frac{1}{\sqrt{8}} \mathbb{Z}^{24}.$$

For integral representatives

$$x \in \sqrt{8} \Lambda_{24},$$

the Leech inner product is

$$\langle x, y \rangle_{\Lambda} = \frac{1}{8} x \cdot y.$$

A minimal vector has squared norm 4, hence an integral representative satisfies

$$\sum_{i=1}^{24} x_i^2 = 32.$$

The Construction-A description uses the extended binary Golay code

$$\mathcal{G}_{24} \subset \mathbb{F}_2^{24}.$$

In particular, an integral representative may be characterized by the usual Golay congruence conditions

$$x_i \equiv 2c_i \pmod{4}, \quad c = (c_1, \dots, c_{24}) \in \mathcal{G}_{24},$$

together with the corresponding coordinate-sum condition.

We use 0-indexed coordinates throughout.

Fix the Golay octad

$$\mathcal{O}_0 = \{1, 12, 13, 14, 16, 17, 18, 22\}.$$

It will be important that this octad is not subsequently identified with the union of two MOG tetrads. Instead, we choose a sextet for which it has intersection pattern

$$(2, 2, 2, 2, 0, 0).$$

3 Explicit Embedding of $\sqrt{2}D_4$

Let e_i denote the standard coordinate vector of $\mathbb{R}^{24}$. Define four vectors in $\mathbb{Z}^{24}$ by

$$v_1 = -4e_1 - 4e_{12}, \tag{1}$$

$$v_2 = 2e_1 + 2e_{12} + 2e_{13} + 2e_{14} + 2e_{16} + 2e_{17} + 2e_{18} + 2e_{22}, \tag{2}$$

$$v_3 = -4e_{13} - 4e_{14}, \tag{3}$$

$$v_4 = -4e_{16} - 4e_{17}. \tag{4}$$

These vectors are understood in the Construction-A coordinate frame, so their Leech norms are computed using the factor 1/8.

Define the 24×4 matrix

$$V = \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix}.$$

3.1 Membership in the Leech Minimal Shell

For v_1, v_3, v_4 ,

$$\|v_i\|^2 = \frac{1}{8}(16 + 16) = 4.$$

For v_2 ,

$$\|v_2\|^2 = \frac{1}{8}(8 \cdot 4) = 4.$$

The vectors v_1, v_3, v_4 are divisible by 4, and hence satisfy the zero-codeword parity condition. Their coordinate sums are

$$-8 \equiv 0 \pmod{8}.$$

For v_2 ,

$$\frac{v_2}{2} \pmod{2} = \mathbf{1}_{\mathcal{O}_0},$$

and since $\mathcal{O}_0$ is a Golay octad,

$$\mathbf{1}_{\mathcal{O}_0} \in \mathcal{G}_{24}.$$

Moreover,

$$\sum_i (v_2)_i = 16 \equiv 0 \pmod{8}.$$

Thus all four vectors are minimal Leech vectors:

$$v_1, v_2, v_3, v_4 \in S_{24}.$$

3.2 The Exact Gram Matrix

The nonzero pairings are

$$\langle v_1, v_2 \rangle = \frac{1}{8}((-4)(2) + (-4)(2)) = -2, \tag{5}$$

$$\langle v_2, v_3 \rangle = \frac{1}{8}((2)(-4) + (2)(-4)) = -2, \tag{6}$$

$$\langle v_2, v_4 \rangle = \frac{1}{8}((2)(-4) + (2)(-4)) = -2. \tag{7}$$

All other off-diagonal pairings vanish.

Therefore

$$V^T V = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & -2 \\ 0 & -2 & 4 & 0 \\ 0 & -2 & 0 & 4 \end{pmatrix}.$$

With the standard D_4 Cartan/Gram matrix

$$G_{D_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix},$$

we obtain the exact identity

$$\boxed{V^T V = 2G_{D_4}}.$$

Hence V realizes an isometric copy of $\sqrt{2}D_4$ inside Λ_{24} .

4 The Four Distinguished Duads

Define

$$D_1 = \{1, 12\}, \quad (8)$$

$$D_2 = \{13, 14\}, \quad (9)$$

$$D_3 = \{16, 17\}, \quad (10)$$

$$D'_4 = \{18, 22\}. \quad (11)$$

The octad is the disjoint union

$$\mathcal{O}_0 = D_1 \sqcup D_2 \sqcup D_3 \sqcup D'_4.$$

For convenience write

$$e_{D_k} = \sum_{i \in D_k} e_i.$$

Then

$$v_1 = -4e_{D_1}, \quad (12)$$

$$v_2 = 2(e_{D_1} + e_{D_2} + e_{D_3} + e_{D'_4}), \quad (13)$$

$$v_3 = -4e_{D_2}, \quad (14)$$

$$v_4 = -4e_{D_3}. \quad (15)$$

The fourth Type-2 root generated by the D_4 root system is

$$r_4 = v_1 + 2v_2 + v_3 + v_4.$$

Substitution gives

$$r_4 = -4e_{D_1} + 4(e_{D_1} + e_{D_2} + e_{D_3} + e_{D'_4}) - 4e_{D_2} - 4e_{D_3} \quad (16)$$

$$= 4e_{D'_4}. \quad (17)$$

Thus the four Type-2 root directions are precisely the four duad directions:

$$\pm 4e_{D_1}, \quad \pm 4e_{D_2}, \quad \pm 4e_{D_3}, \quad \pm 4e_{D'_4}.$$

5 The MOG Sextet

Choose the following six tetrads:

$$B_1 = \{0, 1, 12, 21\}, \quad (18)$$

$$B_2 = \{4, 11, 13, 14\}, \quad (19)$$

$$B_3 = \{5, 6, 16, 17\}, \quad (20)$$

$$B_4 = \{8, 9, 18, 22\}, \quad (21)$$

$$B_5 = \{15, 19, 20, 23\}, \quad (22)$$

$$B_6 = \{2, 3, 7, 10\}. \quad (23)$$

They partition the 24 coordinates:

$$\{0, \dots, 23\} = B_1 \sqcup \dots \sqcup B_6.$$

Their intersections with $\mathcal{O}_0$ are

$$B_1 \cap \mathcal{O}_0 = \{1, 12\} = D_1, \quad (24)$$

$$B_2 \cap \mathcal{O}_0 = \{13, 14\} = D_2, \quad (25)$$

$$B_3 \cap \mathcal{O}_0 = \{16, 17\} = D_3, \quad (26)$$

$$B_4 \cap \mathcal{O}_0 = \{18, 22\} = D'_4, \quad (27)$$

$$B_5 \cap \mathcal{O}_0 = \emptyset, \quad (28)$$

$$B_6 \cap \mathcal{O}_0 = \emptyset. \quad (29)$$

Thus the intersection pattern is

$$\boxed{(2, 2, 2, 2, 0, 0)}.$$

The six tetrads form a Golay sextet: every pairwise union

$$B_i \cup B_j$$

is an octad of $\mathcal{G}_{24}$.

6 Why the Union-of-Tetrads Normalization Fails

The choice

$$\mathcal{O}_0 = B_1 \sqcup B_2$$

would force every vector supported on $\mathcal{O}_0$ to have zero coordinates in $B_3, \dots, B_6$.

The Hexacode

$$\mathcal{H}_6 \subset \mathbb{F}_4^6$$

is an $[6, 3, 4]_{\mathbb{F}_4}$ MDS code. Hence its minimum distance is 4, and there are no nonzero codewords supported on fewer than four coordinates.

Consequently, if a linear coordinate reduction produced

$$\rho(x) = (\rho_1(x), \rho_2(x), 0, 0, 0, 0) \in \mathcal{H}_6,$$

then necessarily

$$\rho(x) = 0.$$

Thus the union-of-two-tetrads normalization is incompatible with a nontrivial Hexacode image for the present subsystem.

The pattern

$$(2, 2, 2, 2, 0, 0)$$

avoids this collapse by distributing the four distinguished duads over four different tetrads.

7 The 24 Root Vectors

The D_4 root shell contains 24 roots. Under the explicit embedding V , every root remains a minimal Leech vector.

They split into three sets of eight.

7.1 Vector Sector

Define

$$R_1 = \{\pm 4e_{D_k} : 1 \leq k \leq 4\}.$$

Thus

$$|R_1| = 8.$$

These are the Type-2 vectors of shape

$$4^2 0^{22}.$$

7.2 First Spinor Sector

Define

$$R_2 = \left\{ 2 \sum_{k=1}^4 s_k e_{D_k} : s_k \in \{+1, -1\}, \prod_{k=1}^4 s_k = +1 \right\}.$$

There are 8 such vectors.

They have shape

$$2^8 0^{16}.$$

The central vector

$$v_2 = 2(e_{D_1} + e_{D_2} + e_{D_3} + e_{D_4})$$

belongs to R_2 .

7.3 Second Spinor Sector

Define

$$R_3 = \left\{ 2 \sum_{k=1}^4 s_k e_{D_k} : s_k \in \{+1, -1\}, \prod_{k=1}^4 s_k = -1 \right\}.$$

Again

$$|R_3| = 8.$$

Thus

$$R(D_4) = R_1 \sqcup R_2 \sqcup R_3.$$

8 Exact Metric Triality

The three sectors have identical internal metric structure.

Proposition 8.1 (Metric Triality). *For each $a \in \{1, 2, 3\}$, the 28 unordered pairs of distinct elements of R_a consist of*

$$4 \text{ pairs with inner product } -4$$

and

$$24 \text{ pairs with inner product } 0.$$

For every pair of distinct sectors R_a, R_b , the 64 cross-sector pairs consist of

$$32 \text{ pairs with inner product } +2$$

and

$$32 \text{ pairs with inner product } -2.$$

Proof. Within R_1 , each vector has one antipode and is orthogonal to the remaining six vectors. Hence there are four antipodal unordered pairs and 24 orthogonal pairs.

For R_2 and R_3 , changing all four signs produces the antipode. Distinct non-antipodal vectors have sign differences in an even number of positions within the same parity sector. The corresponding inner products are zero.

Between two different parity sectors, the sign vectors differ in an odd number of positions, producing inner products $+2$ or -2 . Exactly half of the 64 pairs have each sign. $\square$

The total number of unordered pairs is

$$\binom{24}{2} = 276.$$

The sector decomposition accounts for

$$3\binom{8}{2} = 84$$

intra-sector pairs and

$$3(8 \cdot 8) = 192$$

inter-sector pairs, giving

$$84 + 192 = 276.$$

This is the metric realization of the triality symmetry

$$\boxed{\text{Out}(D_4) \cong S_3}$$

permuting the three eight-element sectors.

9 The Type-Sensitive Shell Reduction

A subtlety arises if one attempts to interpret the MOG reduction as a single global linear map on the lattice.

Consider

$$r_4 = v_1 + 2v_2 + v_3 + v_4.$$

Modulo 2,

$$2v_2 \equiv 0.$$

A naïve global $\mathbb{F}_2$ -linear reduction would therefore imply

$$\rho(r_4) = \rho(v_1) + \rho(v_3) + \rho(v_4).$$

But geometrically r_4 is supported exclusively on D'_4 , whereas the three summands are supported on the first three duads. Therefore the ordinary coordinate-support interpretation cannot simultaneously be a global linear map on simple-root coefficients.

The correct object is a shell-level, type-sensitive map

$$\boxed{\bar{\rho} : R(D_4) \longrightarrow \mathbb{F}_2^4 \times \mathbb{F}_2.}$$

It records the MOG shell data without claiming a linear extension to the whole lattice.

For $r \in R_1$, define

$$\bar{\rho}(\pm 4e_{D_k}) = (\mathbf{e}_k, 0),$$

where $\mathbf{e}_k$ is the k -th standard basis vector of $\mathbb{F}_2^4$.

For

$$r = 2 \sum_{k=1}^4 s_k e_{D_k}, \quad s_k \in \{+1, -1\},$$

define

$$b_k = \frac{1 - s_k}{2} \in \mathbb{F}_2$$

and

$$\bar{\rho}(r) = ((b_1, b_2, b_3, b_4), 1).$$

Thus:

$$R_1 \longrightarrow \{e_1, e_2, e_3, e_4\} \times \{0\}, \quad (30)$$

$$R_2 \longrightarrow \{b \in \mathbb{F}_2^4 : \sum b_k = 0\} \times \{1\}, \quad (31)$$

$$R_3 \longrightarrow \{b \in \mathbb{F}_2^4 : \sum b_k = 1\} \times \{1\}. \quad (32)$$

The first map is 2-to-1, because r and $-r$ have the same shell label. The spinor-sector sign patterns are recorded individually.

10 Root Images in the Ambient MOG Space

Choose an identification

$$\mathbb{F}_2^2 \cong \mathbb{F}_4$$

for each MOG tetrad, together with the standard MOG coordinate convention.

With the duads D_1, D_2, D_3, D'_4 aligned with the first four tetrads, the simple-root shell images may be represented by

$$H_1 = (1, 0, 0, 0, 0, 0), \quad (33)$$

$$H_2 = (1, 1, 1, 1, 0, 0), \quad (34)$$

$$H_3 = (0, 1, 0, 0, 0, 0), \quad (35)$$

$$H_4 = (0, 0, 1, 0, 0, 0). \quad (36)$$

The fourth Type-2 root satisfies

$$r_4 = v_1 + 2v_2 + v_3 + v_4$$

and is supported on D'_4 . Its corresponding shell label is

$$H'_4 = (0, 0, 0, 1, 0, 0).$$

Thus the four Type-2 directions occupy the four coordinate axes of the active four-column subspace.

The central Type-1 root is

$$H_2 = (1, 1, 1, 1, 0, 0).$$

The ambient span is therefore

$$V_\rho = \text{span}_{\mathbb{F}_2}\{H_1, H_2, H_3, H'_4\} = \mathbb{F}_2^4 \times \{0\}^2.$$

It has dimension 4 and contains 16 elements.

Its weight enumerator is

$$\boxed{W_{V_\rho}(z) = 1 + 4z + 6z^2 + 4z^3 + z^4 = (1 + z)^4.}$$

11 The 12 Nonzero Root Images

Under the binary shell labeling, opposite roots have the same image. Hence the 24 roots produce 12 distinct nonzero labels.

Their weights are:

Hamming weight	Number of labels	Number of roots
1	4	8
2	3	6
3	4	8
4	1	2

Thus

$$8 + 6 + 8 + 2 = 24.$$

Inside the 15 nonzero vectors of

$$\mathbb{F}_2^4 \times \{0\}^2,$$

there remain exactly three labels not represented by a D_4 root:

$$u_1 = (1, 1, 0, 0, 0, 0), \tag{37}$$

$$u_2 = (1, 0, 1, 0, 0, 0), \tag{38}$$

$$u_3 = (0, 1, 1, 0, 0, 0). \tag{39}$$

They satisfy

$$u_1 + u_2 = u_3.$$

Consequently

$$V_3 = \{0, u_1, u_2, u_3\} \cong \mathbb{F}_2^2.$$

With respect to the binary dot product,

$$\langle u_i, u_i \rangle = 0,$$

while

$$\langle u_i, u_j \rangle = 1 \quad (i \neq j).$$

This gives the intrinsic three-point symplectic core associated with the triality complement.

12 Triality and the S_3 Action

The three sectors

$$R_1, \quad R_2, \quad R_3$$

are the vector and two half-spinor sectors of D_4 .

The outer automorphism group

$$\text{Out}(D_4) \cong S_3$$

permutes these sectors.

At the combinatorial level, a transposition of the two spinor sectors is represented by changing the sign of one duad. Such a transformation preserves the vector sector while interchanging even and odd sign parity.

A three-cycle can be represented, after choosing the standard four-dimensional realization, by the corresponding Hadamard-type transformation

$$H_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

The role of this matrix here is representation-theoretic: it expresses the familiar triality interchange between the vector and spinor descriptions. The present construction does not require an explicit element of Co_0 realizing every outer automorphism.

13 Interaction with the Hexacode

The Hexacode is the $[6, 3, 4]_{\mathbb{F}_4}$ MDS code

$$\mathcal{H}_6 \subset \mathbb{F}_4^6.$$

Its weight distribution is

$$W_{\mathcal{H}_6}(z) = 1 + 45z^4 + 18z^6.$$

In particular, there are no nonzero codewords of weights 1, 2, 3.

13.1 The Spinor Sectors

Every vector in $R_2 \cup R_3$ has entries ± 2 on the same octad $\mathcal{O}_0$. Modulo 2, the signs disappear:

$$+2 \equiv -2 \equiv 0 \pmod{2}$$

at the level of the doubled lattice representative, while after the standard Construction-A normalization the underlying binary support is the characteristic vector of $\mathcal{O}_0$.

Because

$$|\mathcal{O}_0 \cap B_i| = \begin{cases} 2, & 1 \leq i \leq 4, \\ 0, & i = 5, 6, \end{cases}$$

its MOG image is a weight-four Hexacode word.

With the chosen coordinate convention,

$$\rho(v_2) = (1, 1, 1, 1, 0, 0) \in \mathcal{H}_6.$$

The scalar multiples give the line

$$L_{\mathcal{H}_6} = \mathbb{F}_4(1, 1, 1, 1, 0, 0) \subset \mathcal{H}_6.$$

13.2 The Type-2 Sector

The vectors in R_1 have entries in $4\mathbb{Z}$. Consequently their ordinary binary Construction-A image is zero.

This is why the type-sensitive shell reduction and the ordinary Golay projection must not be conflated.

The shell reduction retains the four transverse duad directions, while the binary Golay reduction sends them into the parity kernel.

Thus:

$$\text{shell information} \neq \text{Golay information.}$$

14 Hexacode Complementarity

Let

$$W = \text{span}_{\mathbb{F}_4}(\rho(D_4)) = \mathbb{F}_4^4 \times \{0\}^2.$$

Then

$$\dim_{\mathbb{F}_4} W = 4.$$

Since

$$\dim_{\mathbb{F}_4} \mathcal{H}_6 = 3$$

and

$$W \cap \mathcal{H}_6 = \mathbb{F}_4(1, 1, 1, 1, 0, 0),$$

we have

$$\dim_{\mathbb{F}_4}(W \cap \mathcal{H}_6) = 1.$$

Therefore

$$\dim_{\mathbb{F}_4}(W + \mathcal{H}_6) = 4 + 3 - 1 = 6.$$

Since

$$\dim_{\mathbb{F}_4} \mathbb{F}_4^6 = 6,$$

it follows that

$$\boxed{W + \mathcal{H}_6 = \mathbb{F}_4^6.}$$

Hence the embedded D_4 coordinate subsystem and the Hexacode are complementary in the ambient MOG space, with precisely one common one-dimensional sector:

$$\boxed{W \cap \mathcal{H}_6 = \mathbb{F}_4 \cdot \rho(v_2).}$$

This is the intrinsic algebraic bridge between the D_4 subsystem and the Hexacode.

15 What the Construction Establishes

The construction establishes the following chain of explicit objects:

$$\boxed{\sqrt{2}D_4 \hookrightarrow \Lambda_{24} \supset S_{24} \longrightarrow \text{MOG sextet} \longrightarrow \mathbb{F}_4^6.}$$

More precisely:

1. The explicit matrix V satisfies

$$V^T V = 2G_{D_4}.$$

2. All four columns of V are minimal Leech vectors.
3. The 24 embedded D_4 roots split into three eight-element sectors

$$R_1 \sqcup R_2 \sqcup R_3$$

with identical metric laws.

4. The three sectors realize the vector and two spinor representations associated with D_4 triality.
5. The MOG sextet

$$(B_1, \dots, B_6)$$

places the four distinguished duads in four distinct tetrads.

6. The shell reduction is explicitly type-sensitive and is not claimed to extend linearly to the entire lattice.
7. The central Type-1 sector determines the nonzero Hexacode line

$$\mathbb{F}_4(1, 1, 1, 1, 0, 0).$$

8. The Type-2 sector supplies the four transverse ambient coordinate directions.
9. The resulting ambient D_4 span and Hexacode satisfy

$$W + \mathcal{H}_6 = \mathbb{F}_4^6, \quad W \cap \mathcal{H}_6 = \mathbb{F}_4 \rho(v_2).$$

Thus the bridge is intrinsic: it uses only the lattice geometry, Construction-A/Golay structure, MOG coordinates, finite-field representation theory, and the D_4 triality decomposition.

16 Epistemic Status

For clarity, the claims of this document have the following status.

Claim	Status	Evidence
Explicit D_4 -type sublattice in Λ_{24}	Verified	Explicit V and exact Gram matrix
$V^T V = 2G_{D_4}$	Verified	Direct coordinate calculation
Explicit compatible MOG sextet	Verified	Six tetrads with required intersections
$(2, 2, 2, 2, 0, 0)$ octad pattern	Verified	Direct set intersection
Three eight-element root sectors	Verified	Explicit root-shell decomposition
Metric triality distributions	Verified	Direct inner-product enumeration
Type-sensitive shell reduction	Explicit construction	Definition on $R(D_4)$
Global linear reduction of the shell map	Not claimed	Explicitly obstructed by r_4
Hexacode weight-four line	Verified under stated MOG conventions	Explicit coordinate realization
Complementarity $W + \mathcal{H}_6 = \mathbb{F}_4^6$	Verified under stated realization	Dimension and intersection calculation
External elliptic-curve/arithmetic bridge	Not part of the intrinsic theory	See Appendix A

17 Conclusion

The essential mathematical bridge is now completely explicit.

Starting from four concrete minimal vectors in the Leech lattice, we obtain a scaled D_4 root system whose 24 roots split naturally into the three triality sectors

$$8_v, \quad 8_s, \quad 8_c.$$

A compatible MOG sextet distributes the four distinguished duads across four tetrads, preventing the otherwise unavoidable Hexacode rank collapse. The resulting shell reduction retains the distinction between the vector and spinor sectors, while the ordinary Construction-A/Golay reduction identifies the two spinor sectors through their common octad support.

The final ambient-space relation

$$\boxed{\text{span}_{\mathbb{F}_4}(\rho(D_4)) + \mathcal{H}_6 = \mathbb{F}_4^6, \quad \text{span}_{\mathbb{F}_4}(\rho(D_4)) \cap \mathcal{H}_6 = \mathbb{F}_4(1, 1, 1, 1, 0, 0)}$$

is therefore an intrinsic finite-field statement attached to the chosen MOG realization.

No external numerical modulus is required to obtain this structure.

A Audited Numerical Discretization Attempt

The original project also considered a separate numerical construction involving a large prime P , decimal truncations of analytic constants, and an elliptic curve

$$E_P : y^2 = x^3 + Ax + B \pmod{P}.$$

That construction is logically independent of the lattice–MOG–Hexacode bridge developed above.

The relevant numerical claims were audited separately. In particular, the proposed identities

$$B = 2A, \quad j(E) = j(-163), \quad a_P = 3$$

do not hold for the published parameters.

Likewise, the proposed CM relation

$$4P = 9 + 163y^2$$

is incompatible with the published prime.

The important methodological conclusion is therefore not that the intrinsic lattice/coding construction fails, but rather that the numerical elliptic-curve layer does not follow from it.

Accordingly, this appendix is deliberately separated from the main construction.

A.1 The CM Congruence Obstruction

If

$$4P = 9 + 163y^2,$$

then necessarily

$$4P - 9 \equiv 0 \pmod{163}.$$

For the published value of P , this congruence fails. Thus the proposed discriminant-163 lock is not an identity satisfied by the published parameters.

A.2 The Coefficient Lock

The claimed relation

$$B = 2A$$

also fails for the published coefficients. Hence it cannot be used as an intrinsic algebraic consequence of the D_4 , Leech, or Hexacode construction.

A.3 The Frobenius Trace

The proposed Frobenius trace

$$a_P = 3$$

and the consequent point-count identity

$$\#E(\mathbb{F}_P) = P - 2$$

are likewise separate arithmetic assertions.

Their verification requires an independent finite-field computation and does not follow from the MOG or Hexacode data.

A.4 Interpretation

The numerical layer should therefore be regarded as an independent discretization experiment rather than as a theorem arising from the intrinsic bridge.

The main mathematical result of this document is the coordinate-explicit chain

$$\boxed{\sqrt{2}D_4 \hookrightarrow \Lambda_{24} \longrightarrow \text{Golay/MOG data} \longrightarrow \mathbb{F}_4^6,}$$

together with the exact triality and Hexacode complementarity statements.