

Orthogonal Frames and Maximal Orthogonal Subsystems in the Exceptional Root Systems E_6 , E_7 , and E_8

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Abstract

We study ordered orthogonal frames of roots and maximal orthogonal subsystems in the exceptional simply-laced root systems E_6 , E_7 , and E_8 . The central structural phenomenon is that sufficiently large orthogonal frames exhibit unique completion rigidity.

For E_7 , we construct a canonical projection from the root lattice modulo 2 onto a six-dimensional symplectic vector space over $\mathbb{F}_2$. This map identifies the 63 root lines with the nonzero vectors of $\mathbb{F}_2^6$, and Euclidean orthogonality with symplectic orthogonality. Consequently, maximal orthogonal subsystems of type A_1^7 correspond bijectively to Lagrangian subspaces, of which there are exactly 135. By a dimension-cardinality argument, any orthogonal set of $k \geq 4$ root lines spans a full 3-dimensional Lagrangian subspace, establishing that every orthogonal frame of size at least four has a unique maximal completion in E_7 .

For E_8 , the orthogonal complement of every root line is isomorphic to E_7 . Slicing and incidence double-counting prove that E_8 contains exactly 2025 maximal subsystems of type A_1^8 . By reducing intersections through a common root line to the E_7 Lagrangian geometry, we prove that any two distinct A_1^8 subsystems share at most four root lines. This establishes that every orthogonal frame of size at least five has a unique maximal A_1^8 completion in E_8 .

These completion theorems yield exact, closed-form enumerations for all higher frame counts:

$$N_k(E_7) = 135 \cdot 2^k \cdot 7^k \quad (4 \leq k \leq 7)$$

and

$$N_k(E_8) = 2025 \cdot 2^k \cdot 8^k \quad (5 \leq k \leq 8).$$

Furthermore, single-root slicing establishes the exact reduction

$$N_k(E_8) = 240 N_{k-1}(E_7) \quad (1 \leq k \leq 8).$$

We show that the two closed expressions agree for every $k \in \{5, 6, 7, 8\}$, and that this agreement reflects the geometric compatibility between the ambient A_1^8 cross-polytopal decomposition of E_8 and its E_7 root-complement slicing.

Contents

1	Introduction	2
1.1	Background and motivation	2
1.2	Summary of results	3
1.3	Distinction from standard background	4
2	Root lines and frame enumeration	4

3	The complete sequence for E_6	5
4	The symplectic model of E_7	6
4.1	Construction from the root lattice modulo 2	6
4.2	The root-line bijection	7
5	Lagrangian subspaces and the 135 maximal systems	9
6	Intersections of A_1^7 subsystems	10
7	Unique completion in E_7	11
8	Frame enumeration in E_7	11
8.1	Complete E_7 spectrum	12
9	The E_8 slice correspondence	13
10	Counting maximal A_1^8 subsystems in E_8	13
11	Intersections of maximal A_1^8 systems	14
12	Unique completion in E_8	14
13	High-order frame enumeration in E_8	15
14	Single-root slicing and compatibility	15
15	The complete E_8 spectrum	16
16	Unified master spectrum	17
17	Structural interpretation	17
17.1	E_6	17
17.2	E_7	18
17.3	E_8	18
18	Frame size and Siegel genus	18
19	Further questions	19
20	Conclusion	19

1 Introduction

1.1 Background and motivation

Let R be an irreducible, simply-laced root system in a Euclidean vector space $(E, (\cdot, \cdot))$, normalized so that

$$(r, r) = 2 \quad \text{for all } r \in R.$$

An ordered orthogonal k -frame in R is an ordered sequence

$$(r_1, \dots, r_k) \in R^k$$

satisfying

$$(r_i, r_j) = 2\delta_{ij}.$$

We denote by $N_k(R)$ the number of such ordered orthogonal k -frames.

For the infinite classical families A_n and D_n , uniform combinatorial formulas for $N_k(R)$ are well understood. In contrast, for the exceptional root systems E_6 , E_7 , and E_8 , one naturally encounters a sequence of changing orthogonal complements. This permits direct computation of small frame counts, but obscures the global geometry of large orthogonal systems.

The purpose of this paper is to make the high-order structure explicit. The main phenomenon is a rigidity statement: sufficiently large orthogonal systems determine their maximal orthogonal completion uniquely.

1.2 Summary of results

To avoid ambiguities involving signs, we work primarily with *root lines*

$$L = \{\pm r\}.$$

These correspond to one-dimensional root sublattices of type A_1 . A maximal orthogonal collection of root lines therefore determines a subsystem of type A_1^m .

The main results are as follows.

1. **The symplectic model of E_7 .** We construct a canonical quotient

$$Q(E_7)/2Q(E_7) \longrightarrow \mathbb{F}_2^6$$

whose kernel is the radical of the induced alternating form. The resulting symplectic space identifies the 63 root lines of E_7 with the 63 nonzero vectors of $\mathbb{F}_2^6$.

2. **Lagrangian classification.** The A_1^7 subsystems of E_7 correspond bijectively to the 3-dimensional totally isotropic subspaces of $\mathbb{F}_2^6$. There are exactly

$$135$$

such Lagrangians.

3. **Unique completion in E_7 .** Every orthogonal set of at least four root lines spans a Lagrangian, and hence determines a unique A_1^7 subsystem.
4. **E_8 slicing.** The orthogonal complement of every root line in E_8 is an E_7 root system. Incidence double-counting therefore gives exactly

$$2025$$

maximal A_1^8 subsystems in E_8 .

5. **Unique completion in E_8 .** Any two distinct A_1^8 subsystems share at most four root lines. Consequently, every orthogonal system of at least five root lines is contained in a unique A_1^8 subsystem.

6. **Closed frame formulas.** For the high-order ranges,

$$N_k(E_7) = 135 \cdot 2^k \cdot 7^{\underline{k}} \quad (4 \leq k \leq 7),$$

and

$$N_k(E_8) = 2025 \cdot 2^k \cdot 8^{\underline{k}} \quad (5 \leq k \leq 8).$$

7. **Compatibility with slicing.** For every

$$k \in \{5, 6, 7, 8\},$$

the two descriptions of $N_k(E_8)$ agree:

$$N_k(E_8) = 240 N_{k-1}(E_7) = 2025 \cdot 2^k \cdot 8^{\underline{k}}.$$

1.3 Distinction from standard background

We separate standard Lie-theoretic facts from the arguments developed here.

- *Classical background:* the root counts $|E_6| = 72$, $|E_7| = 126$, $|E_8| = 240$; the root-complement identifications; the standard Weyl-group facts; the isomorphism

$$W(E_7)/\{\pm I\} \cong \mathrm{Sp}(6, \mathbb{F}_2);$$

and the existence of the maximal A_1^7 and A_1^8 systems.

- *Arguments developed here:* the root-lattice construction of the symplectic quotient; the dimension-cardinality proof of unique completion in E_7 ; the incidence derivation of the 2025 count in E_8 ; the intersection bound for A_1^8 systems; the resulting unique completion theorem in E_8 ; and the closed high-order frame formulas.

2 Root lines and frame enumeration

Definition 2.1. Let R be a simply-laced root system. A *root line* of R is a pair of opposite roots

$$L = \{\pm r\} \subset R.$$

We denote the set of root lines by $\mathcal{L}(R)$. Thus

$$|\mathcal{L}(R)| = \frac{|R|}{2}.$$

Two root lines

$$L_1 = \{\pm r_1\}, \quad L_2 = \{\pm r_2\}$$

are orthogonal if

$$(r_1, r_2) = 0.$$

This is independent of the choices of representatives.

Definition 2.2. An *orthogonal k -line system* is a set

$$\{L_1, \dots, L_k\} \subseteq \mathcal{L}(R)$$

of k pairwise orthogonal root lines. We denote the number of unordered orthogonal k -line systems by $M_k(R)$.

Lemma 2.3. *For every simply-laced root system R ,*

$$N_k(R) = 2^k k! M_k(R),$$

or equivalently,

$$M_k(R) = \frac{N_k(R)}{2^k k!}.$$

Proof. Given an unordered set of k mutually orthogonal root lines, there are $k!$ ways to order them and two choices of sign for each line. Thus every unordered system produces exactly $2^k k!$ ordered frames. $\square$

Definition 2.4. A root subsystem $S \subseteq R$ is of type A_1^m if it consists of m mutually orthogonal root lines together with their opposite roots.

If R has rank n , then

$$M_k(R) = N_k(R) = 0 \quad (k > n).$$

3 The complete sequence for E_6

Lemma 3.1. *The maximum size of an orthogonal set of root lines in E_6 is 4.*

Proof. Suppose there were five mutually orthogonal root lines in E_6 . They would generate an A_1^5 subsystem of rank 5. The classification of rank-5 subsystems in E_6 gives types A_5 , D_5 , and $A_4 \oplus A_1$, none of which contains A_1^5 . Thus no such set exists. $\square$

Proposition 3.2. *For $1 \leq k \leq 4$,*

$$N_k(E_6) = \prod_{j=1}^k d_j, \quad (d_1, d_2, d_3, d_4) = (72, 30, 12, 2).$$

Hence

k	1	2	3	4	≥ 5
$N_k(E_6)$	72	2 160	25 920	51 840	0
$M_k(E_6)$	36	270	540	135	0

Proof. There are 72 choices for the first root. The orthogonal complement of a root in E_6 is A_5 , containing 30 roots.

Inside A_5 , the orthogonal complement of a root is A_3 , containing 12 roots. The orthogonal complement of a root in A_3 is A_1 , containing 2 roots. Thus

$$N_1 = 72, \quad N_2 = 72 \cdot 30, \quad N_3 = 72 \cdot 30 \cdot 12, \quad N_4 = 72 \cdot 30 \cdot 12 \cdot 2.$$

The values of M_k follow from [lemma 2.3](#). $\square$

Remark 3.3. *The value*

$$M_4(E_6) = 135$$

counts the maximal A_1^4 systems in E_6 .

4 The symplectic model of E_7

The exceptional root system E_7 has 126 roots and therefore 63 root lines. We now construct its symplectic model directly from the root lattice.

4.1 Construction from the root lattice modulo 2

Let

$$Q = Q(E_7) = \bigoplus_{i=1}^7 \mathbb{Z}\alpha_i$$

be the root lattice. Let $P = Q^*$ be its dual lattice. Since

$$[P : Q] = 2,$$

we have

$$\det(Q) = 2.$$

Set

$$\overline{Q} = Q/2Q \cong \mathbb{F}_2^7.$$

The root-lattice form induces

$$\bar{b} : \overline{Q} \times \overline{Q} \longrightarrow \mathbb{F}_2, \quad \bar{b}(x + 2Q, y + 2Q) = (x, y) \pmod{2}.$$

Lemma 4.1. *The form $\bar{b}$ is alternating and has a one-dimensional radical.*

Proof. Since Q is even,

$$(x, x) \in 2\mathbb{Z}$$

for all $x \in Q$. Hence

$$\bar{b}(u, u) = 0$$

for all $u \in \overline{Q}$, so $\bar{b}$ is alternating.

Now $x + 2Q$ belongs to the radical precisely when

$$(x, y) \equiv 0 \pmod{2}$$

for all $y \in Q$. Equivalently,

$$x/2 \in Q^* = P.$$

Thus

$$\text{Rad}(\bar{b}) = (2P \cap Q)/2Q = 2P/2Q \cong P/Q.$$

Since $[P : Q] = 2$, the radical has dimension one over $\mathbb{F}_2$. □

Let

$$u_0 \in \overline{Q} \setminus \{0\}$$

denote the unique nonzero radical vector. In terms of fundamental weights, one may take

$$u_0 = 2\varpi_7 + 2Q.$$

Definition 4.2. The *canonical symplectic space* of E_7 is

$$V = \overline{Q} / \text{Rad}(\overline{b}).$$

Thus

$$V \cong \mathbb{F}_2^6.$$

The form $\overline{b}$ descends to a nondegenerate alternating form

$$\omega : V \times V \longrightarrow \mathbb{F}_2.$$

Consequently,

$$(V, \omega) \cong (\mathbb{F}_2^6, \omega)$$

is a six-dimensional symplectic vector space.

The symplectic group has order

$$|\text{Sp}(6, \mathbb{F}_2)| = 2^9(2^2 - 1)(2^4 - 1)(2^6 - 1) = 1\,451\,520.$$

The Weyl group satisfies

$$|W(E_7)| = 2\,903\,040$$

and classically

$$W(E_7)/\{\pm I\} \cong \text{Sp}(6, \mathbb{F}_2).$$

4.2 The root-line bijection

Let

$$\pi : Q \longrightarrow V$$

be the quotient map

$$x \longmapsto (x + 2Q) + \text{Rad}(\overline{b}).$$

Proposition 4.3. *The map π has the following properties.*

(i) *For every root $r \in E_7$,*

$$\pi(r) \neq 0.$$

(ii) *For roots $r_1, r_2 \in E_7$,*

$$\pi(r_1) = \pi(r_2) \iff r_2 = \pm r_1.$$

(iii) *The induced map*

$$\phi : \mathcal{L}(E_7) \longrightarrow V \setminus \{0\}$$

is a bijection.

(iv) *For root lines L_1, L_2 ,*

$$L_1 \perp L_2 \iff \omega(\phi(L_1), \phi(L_2)) = 0.$$

Proof. For (i), suppose $\pi(r) = 0$. Then

$$r + 2Q \in \text{Rad}(\bar{b}).$$

Since $r \neq 0$, this gives

$$r \in 2P.$$

Hence

$$(r, \alpha) \equiv 0 \pmod{2}$$

for every root α . But every root in an irreducible simply-laced root system of rank at least 2 has a root with inner product ± 1 . This is a contradiction.

For (ii), suppose

$$\pi(r_1) = \pi(r_2).$$

Then

$$r_1 - r_2 \in \text{Rad}(\bar{b}).$$

By definition of the radical,

$$(r_1 - r_2, \alpha) \equiv 0 \pmod{2} \quad \text{for every root } \alpha \in E_7.$$

Taking $\alpha = r_1$ gives

$$2 - (r_1, r_2) \equiv 0 \pmod{2}.$$

Hence (r_1, r_2) is even. Since the inner products of two roots in a simply-laced system belong to

$$\{0, \pm 1, \pm 2\},$$

we obtain

$$(r_1, r_2) \in \{0, \pm 2\}.$$

If

$$(r_1, r_2) = \pm 2,$$

then equality holds in the Cauchy–Schwarz inequality, so

$$r_2 = \pm r_1.$$

It remains to exclude

$$(r_1, r_2) = 0.$$

In that case

$$r_1 \in R_2 := r_2^\perp \cap E_7.$$

The orthogonal complement of a root in E_7 is a root system of type D_6 . Since D_6 is irreducible of rank at least two, every root of D_6 has a root adjacent to it, i.e., there exists

$$\alpha \in R_2$$

such that

$$(r_1, \alpha) = \pm 1.$$

Because $\alpha \in r_2^\perp$,

$$(r_2, \alpha) = 0.$$

Therefore

$$(r_1 - r_2, \alpha) = (r_1, \alpha) - (r_2, \alpha) = \pm 1,$$

which is odd, contradicting

$$(r_1 - r_2, \alpha) \equiv 0 \pmod{2}.$$

Thus $(r_1, r_2) = 0$ is impossible, and

$$r_2 = \pm r_1.$$

For (iii), there are

$$|\mathcal{L}(E_7)| = 63$$

root lines and

$$|V \setminus \{0\}| = 2^6 - 1 = 63$$

nonzero vectors. Part (ii) gives injectivity, hence bijectivity.

For (iv),

$$\omega(\phi(L_1), \phi(L_2)) \equiv (r_1, r_2) \pmod{2}.$$

For distinct root lines, the inner product cannot be ± 2 , so it belongs to $\{0, \pm 1\}$. Hence it is even precisely when it is zero. Thus

$$L_1 \perp L_2 \iff \omega(\phi(L_1), \phi(L_2)) = 0.$$

□

5 Lagrangian subspaces and the 135 maximal systems

Definition 5.1. A subspace $W \subseteq V$ is *totally isotropic* if

$$\omega(u, v) = 0 \quad \text{for all } u, v \in W.$$

Since $\dim V = 6$, the largest possible dimension of a totally isotropic subspace is 3. A 3-dimensional totally isotropic subspace is called a *Lagrangian*.

Proposition 5.2. *Under the bijection ϕ , maximal orthogonal sets of root lines in E_7 correspond bijectively to Lagrangian subspaces of V . Consequently every maximal orthogonal subsystem is of type A_1^7 .*

Proof. Let W be Lagrangian. It contains

$$2^3 - 1 = 7$$

nonzero vectors. By [proposition 4.3](#), these correspond to seven pairwise orthogonal root lines.

Conversely, let $\mathcal{S}$ be a maximal orthogonal set of root lines. Its image under ϕ is pairwise symplectically orthogonal. Let

$$W = \text{Span}_{\mathbb{F}_2}(\phi(\mathcal{S})).$$

Then W is totally isotropic, so

$$\dim W \leq 3.$$

If $\dim W < 3$, then W lies in a Lagrangian W' and

$$\phi^{-1}(W' \setminus \{0\})$$

would be a strictly larger orthogonal system. Thus

$$\dim W = 3.$$

Hence W is Lagrangian and contains exactly seven nonzero vectors.

□

Theorem 5.3. *There are exactly 135 maximal A_1^7 subsystems in E_7 .*

Proof. Count ordered bases of Lagrangian subspaces.

Choose an ordered isotropic basis

$$(u_1, u_2, u_3).$$

There are

$$2^6 - 1 = 63$$

choices for u_1 .

Once u_1 is chosen, u_2 must lie in $u_1^\perp$ but outside $\text{Span}(u_1)$, giving

$$2^5 - 2 = 30$$

choices.

Finally, u_3 must lie in

$$\text{Span}(u_1, u_2)^\perp$$

but outside $\text{Span}(u_1, u_2)$, giving

$$2^4 - 2^2 = 12$$

choices.

Thus there are

$$63 \cdot 30 \cdot 12$$

ordered isotropic bases.

Each Lagrangian has

$$|\text{GL}_3(\mathbb{F}_2)| = (8-1)(8-2)(8-4) = 168$$

ordered bases. Therefore

$$|\text{Lag}(V)| = \frac{63 \cdot 30 \cdot 12}{168} = 135.$$

□

6 Intersections of A_1^7 subsystems

Theorem 6.1. *Let T_1, T_2 be distinct A_1^7 subsystems of E_7 . Then*

$$|T_1 \cap T_2| \leq 3.$$

More precisely,

$$|T_1 \cap T_2| \in \{0, 1, 3\}.$$

Proof. Let W_1, W_2 be the corresponding Lagrangians. Since they are distinct,

$$d := \dim(W_1 \cap W_2) \leq 2.$$

The common root lines correspond to the nonzero vectors of $W_1 \cap W_2$. Hence

$$|T_1 \cap T_2| = 2^d - 1.$$

For $d = 0, 1, 2$ this gives

$$0, \quad 1, \quad 3.$$

□

Remark 6.2. *The Lagrangians are Fano planes $PG(2, \mathbb{F}_2)$ inside the symplectic polar space $W(5, 2)$. Two distinct Lagrangians can therefore meet in the empty set, a point, or a projective line containing three points. In particular, intersection size 2 cannot occur.*

7 Unique completion in E_7

Theorem 7.1 (Unique completion in E_7). *Let*

$$\mathcal{F} = \{L_1, \dots, L_k\}$$

be an orthogonal set of root lines in E_7 . If

$$4 \leq k \leq 7,$$

then $\mathcal{F}$ is contained in a unique A_1^7 subsystem.

Proof. Set

$$S = \phi(\mathcal{F})$$

and

$$W = \text{Span}_{\mathbb{F}_2}(S).$$

Since S is pairwise symplectically orthogonal, W is totally isotropic and therefore

$$\dim W \leq 3.$$

But W contains at least four distinct nonzero vectors. A two-dimensional vector space over $\mathbb{F}_2$ has only

$$2^2 - 1 = 3$$

nonzero vectors. Thus

$$\dim W = 3.$$

Hence W is Lagrangian.

Any A_1^7 subsystem containing $\mathcal{F}$ corresponds to a Lagrangian W_T containing S . Since W_T is a vector space,

$$W \subseteq W_T.$$

Both spaces have dimension 3, so

$$W = W_T.$$

Thus the completion is unique. □

Corollary 7.2. *For every*

$$k \in \{4, 5, 6, 7\},$$

the 135 maximal A_1^7 subsystems partition the set of orthogonal k -line systems in E_7 .

8 Frame enumeration in E_7

Theorem 8.1. *For*

$$4 \leq k \leq 7,$$

we have

$$\boxed{N_k(E_7) = 135 \cdot 2^k \cdot 7^k}.$$

Equivalently,

$$\boxed{M_k(E_7) = 135 \binom{7}{k}}.$$

Proof. Every maximal A_1^7 subsystem contains seven mutually orthogonal root lines. By [corollary 7.2](#), every orthogonal k -line system with $k \geq 4$ belongs to exactly one such subsystem. Therefore

$$M_k(E_7) = 135 \binom{7}{k}.$$

Applying [lemma 2.3](#),

$$N_k(E_7) = 2^k k! 135 \binom{7}{k} = 135 \cdot 2^k \cdot 7^{\underline{k}}.$$

□

8.1 Complete E_7 spectrum

For $k \leq 3$, multiple Lagrangians may contain the same system.

For $k = 1$,

$$N_1(E_7) = 126, \quad M_1(E_7) = 63.$$

For $k = 2$, each root has 30 orthogonal roots, so

$$M_2(E_7) = \frac{63 \cdot 30}{2} = 945,$$

and hence

$$N_2(E_7) = 945 \cdot 2^2 \cdot 2! = 7\,560.$$

For $k = 3$, there are two types of isotropic triples. Dependent triples have the form

$$u + v + w = 0$$

inside a Lagrangian plane. Their number is

$$\frac{63 \cdot 30}{6} = 315.$$

Independent triples are bases of Lagrangians, giving

$$135 \cdot \frac{168}{6} = 3\,780.$$

Thus

$$M_3(E_7) = 315 + 3\,780 = 4\,095$$

and

$$N_3(E_7) = 4\,095 \cdot 2^3 \cdot 3! = 196\,560.$$

Consequently,

k	1	2	3	4	5	6	7	≥ 8
$N_k(E_7)$	126	7 560	196 560	1 814 400	10 886 400	43 545 600	87 091 200	0
$M_k(E_7)$	63	945	4 095	4 725	2 835	945	135	0

9 The E_8 slice correspondence

The root system E_8 has

$$|E_8| = 240$$

roots and

$$|\mathcal{L}(E_8)| = 120$$

root lines.

Lemma 9.1. *For every root line $L \in \mathcal{L}(E_8)$,*

$$L^\perp \cap E_8 \cong E_7.$$

Proof. The Weyl group $W(E_8)$ acts transitively on roots. It is therefore enough to consider one root.

In the standard realization of E_8 , choose

$$r = (1, 1, 0, 0, 0, 0, 0, 0).$$

The roots orthogonal to r form a rank-7 root system. The standard root-complement classification identifies it as E_7 . Since E_8 has 240 roots, the complement has

$$126$$

roots, as required for E_7 . □

10 Counting maximal A_1^8 subsystems in E_8

Theorem 10.1. *The root system E_8 contains exactly*

$$2025$$

maximal subsystems of type A_1^8 .

Proof. Consider

$$\Omega = \{(L, S) : L \in \mathcal{L}(E_8), S \text{ is an } A_1^8 \text{ subsystem, } L \subset S\}.$$

There are 120 root lines. Fix one line L . By [lemma 9.1](#),

$$L^\perp \cap E_8 \cong E_7.$$

An A_1^8 subsystem containing L is obtained by choosing an A_1^7 subsystem in this E_7 complement. There are 135 such systems. Hence

$$|\Omega| = 120 \cdot 135 = 16\,200.$$

On the other hand, each A_1^8 subsystem contains exactly eight root lines, so

$$|\Omega| = 8 \cdot \#\{A_1^8 \subset E_8\}.$$

Therefore

$$\#\{A_1^8 \subset E_8\} = \frac{16\,200}{8} = 2025.$$

□

Remark 10.2. *The value*

$$2025 = 45^2$$

also agrees with the classical coding-theoretic description of the E_8 lattice using the extended binary Hamming code. The double-counting argument above, however, obtains the number directly from the E_7 geometry.

11 Intersections of maximal A_1^8 systems

Theorem 11.1. *Let S_1, S_2 be distinct A_1^8 subsystems of E_8 . Then*

$$|S_1 \cap S_2| \leq 4.$$

Proof. Suppose S_1 and S_2 have a common root line L . Then every other common root line lies in

$$L^\perp \cap E_8 \cong E_7.$$

Removing the common line L , the two A_1^8 systems determine two A_1^7 systems in this E_7 slice.

If $S_1 \neq S_2$, the two induced A_1^7 systems are distinct. By [theorem 6.1](#), they share at most three additional root lines. Therefore

$$|S_1 \cap S_2| \leq 1 + 3 = 4.$$

If the two systems have no common line, the bound is immediate. $\square$

Remark 11.2. *Computational enumeration suggests that intersection size 3 does not occur among distinct maximal A_1^8 systems. This observation is not required for any result in this paper and is not promoted to a theorem. Determining the complete intersection distribution of the 2025 maximal A_1^8 systems is left as an open combinatorial question.*

12 Unique completion in E_8

Theorem 12.1. *Let*

$$\mathcal{F} = \{L_1, \dots, L_k\}$$

be an orthogonal set of root lines in E_8 . If

$$5 \leq k \leq 8,$$

then $\mathcal{F}$ is contained in a unique A_1^8 subsystem.

Proof. Existence follows by extending any orthogonal set to a maximal orthogonal set.

For uniqueness, suppose two distinct A_1^8 systems S_1, S_2 contain $\mathcal{F}$. Then

$$\mathcal{F} \subseteq S_1 \cap S_2.$$

Since $|\mathcal{F}| \geq 5$, this gives

$$|S_1 \cap S_2| \geq 5,$$

contradicting [theorem 11.1](#). Hence the maximal completion is unique. $\square$

Corollary 12.2. *For every*

$$k \in \{5, 6, 7, 8\},$$

the 2025 maximal A_1^8 subsystems partition the set of orthogonal k -line systems in E_8 .

13 High-order frame enumeration in E_8

Theorem 13.1. *For*

$$5 \leq k \leq 8,$$

the number of ordered orthogonal k -frames in E_8 is

$$N_k(E_8) = 2025 \cdot 2^k \cdot 8^k.$$

Equivalently,

$$M_k(E_8) = 2025 \binom{8}{k}.$$

Proof. By [theorem 12.1](#), every orthogonal k -line system with $k \geq 5$ lies in a unique maximal A_1^8 system. Therefore

$$M_k(E_8) = 2025 \binom{8}{k}.$$

Applying [lemma 2.3](#),

$$N_k(E_8) = 2^k k! M_k(E_8) = 2025 \cdot 2^k \cdot 8^k.$$

□

14 Single-root slicing and compatibility

Proposition 14.1. *For every*

$$1 \leq k \leq 8,$$

one has

$$N_k(E_8) = 240 N_{k-1}(E_7).$$

Proof. Choose the first root r_1 of an ordered orthogonal k -frame in E_8 . There are 240 choices.

Once r_1 is chosen, all remaining roots lie in

$$r_1^\perp \cap E_8 \cong E_7.$$

Thus the remaining $k - 1$ entries form an ordered orthogonal $(k - 1)$ -frame in E_7 . Conversely, every such frame together with r_1 produces an ordered orthogonal k -frame in E_8 .

Hence

$$N_k(E_8) = 240 N_{k-1}(E_7).$$

□

Theorem 14.2. *For every*

$$k \in \{5, 6, 7, 8\},$$

the slicing formula and the maximal-completion formula agree:

$$240 N_{k-1}(E_7) = 2025 \cdot 2^k \cdot 8^k.$$

Proof. For

$$k \in \{5, 6, 7, 8\},$$

we have

$$k - 1 \in \{4, 5, 6, 7\}.$$

Applying the E_7 high-order formula gives

$$N_{k-1}(E_7) = 135 \cdot 2^{k-1} \cdot 7^{k-1}.$$

Therefore

$$240 N_{k-1}(E_7) = (240 \cdot 135) 2^{k-1} 7^{k-1} = 32\,400 2^{k-1} 7^{k-1}.$$

On the other hand,

$$8^k = 8 \cdot 7^{k-1}$$

and

$$2^k = 2 \cdot 2^{k-1}.$$

Thus

$$2025 \cdot 2^k \cdot 8^k = 2025 \cdot (2 \cdot 8) 2^{k-1} 7^{k-1}.$$

Since

$$2025 \cdot 16 = 32\,400$$

and

$$240 \cdot 135 = 32\,400,$$

the two expressions are equal for every

$$k \in \{5, 6, 7, 8\}.$$

□

Remark 14.3. *The equality*

$$240 \cdot 135 = 2025 \cdot 16$$

is the numerical compatibility between the 240 roots of E_8 , the 135 maximal A_1^7 systems in an E_7 slice, and the 2025 maximal A_1^8 systems in E_8 . The equality is not merely an arithmetic coincidence: both sides count the same high-order orthogonal-frame structures through two different geometric decompositions.

15 The complete E_8 spectrum

The low-order values can be obtained directly by slicing.

For $k = 1$,

$$N_1(E_8) = 240, \quad M_1(E_8) = 120.$$

For $k = 2$,

$$N_2(E_8) = 240 \cdot 126 = 30\,240.$$

Hence

$$M_2(E_8) = \frac{30\,240}{2^2 \cdot 2!} = 3\,780.$$

For $k = 3$,

$$N_3(E_8) = 240 \cdot 7\,560 = 1\,814\,400,$$

so

$$M_3(E_8) = 37\,800.$$

For $k = 4$,

$$N_4(E_8) = 240 \cdot 196\,560 = 47\,174\,400,$$

and therefore

$$M_4(E_8) = 122\,850.$$

For $k = 5, 6, 7, 8$, the unique-completion formula gives the remaining values.

Thus the complete E_8 spectrum is

k	1	2	3	4	5	6	7	8	≥ 9
$N_k(E_8)$	240	30 240	1 814 400	47 174 400	435 456 000	2 612 736 000	10 450 944 000	20 901 888 000	0
$M_k(E_8)$	120	3 780	37 800	122 850	113 400	56 700	16 200	2 025	0

In particular, the high-order values satisfy

$$N_k(E_8) = 2025 \cdot 2^k \cdot 8^{\underline{k}}$$

for

$$k \in \{5, 6, 7, 8\},$$

and simultaneously

$$N_k(E_8) = 240 N_{k-1}(E_7).$$

16 Unified master spectrum

For convenient comparison, we collect the complete ordered-frame and root-line spectra of all three exceptional systems.

The table satisfies throughout the identity

$$N_k(R) = 2^k k! M_k(R).$$

17 Structural interpretation

The preceding calculations exhibit three distinct regimes.

17.1 E_6

The maximal orthogonal rank is 4, despite the ambient rank being 6. The complete sequence is therefore obtained by a finite chain of orthogonal complements

$$E_6 \supset A_5 \supset A_3 \supset A_1.$$

Table 1: Complete orthogonal frame spectrum $N_k(R)$ and line spectrum $M_k(R)$ for E_6 , E_7 , and E_8 .

k	$N_k(E_6)$	$N_k(E_7)$	$N_k(E_8)$
1	72	126	240
2	2 160	7 560	30 240
3	25 920	196 560	1 814 400
4	51 840	1 814 400	47 174 400
5	0	10 886 400	435 456 000
6	0	43 545 600	2 612 736 000
7	0	87 091 200	10 450 944 000
8	0	0	20 901 888 000
≥ 9	0	0	0

k	$M_k(E_6)$	$M_k(E_7)$	$M_k(E_8)$
1	36	63	120
2	270	945	3 780
3	540	4 095	37 800
4	135	4 725	122 850
5	0	2 835	113 400
6	0	945	56 700
7	0	135	16 200
8	0	0	2 025
≥ 9	0	0	0

17.2 E_7

The E_7 geometry is governed globally by the symplectic space

$$(\mathbb{F}_2^6, \omega).$$

The 63 root lines become the 63 nonzero vectors, and the A_1^7 systems become Lagrangian subspaces.

The threshold $k = 4$ is explained purely by finite-dimensional linear algebra: four distinct nonzero vectors cannot lie in a two-dimensional $\mathbb{F}_2$ -space, while every totally isotropic subspace has dimension at most three.

17.3 E_8

The E_8 geometry inherits this structure through root-complement slicing. Every root line determines an E_7 slice, and every A_1^8 system is obtained by adjoining the distinguished root line to an A_1^7 system in that slice.

The threshold $k = 5$ follows from the intersection bound

$$|S_1 \cap S_2| \leq 4.$$

18 Frame size and Siegel genus

It is important to distinguish two independent parameters that can occur in applications to Siegel modular forms.

The integer

$$k$$

used throughout this paper denotes the *frame size*, namely the number of mutually orthogonal roots in an ordered orthogonal frame.

By contrast, the integer

$$g$$

denotes the *Siegel genus* of an ambient Siegel modular form.

These quantities should not be identified. In particular, the appearance of $N_4(R)$ in genus-four modular-form calculations is a statement about the role of four-fold orthogonal configurations in a particular modular-form setting; it does not mean that the frame size k is itself a genus parameter.

The present paper concerns the root-theoretic quantities

$$N_k(E_6), \quad N_k(E_7), \quad N_k(E_8),$$

independently of any particular Siegel genus.

19 Further questions

Several natural questions remain.

1. Determine the complete intersection distribution of the 2025 maximal A_1^8 subsystems of E_8 . The present argument proves only the upper bound

$$|S_1 \cap S_2| \leq 4$$

for distinct systems. Computational evidence suggests that intersection size 3 may be absent, but this is not proved here.

2. Give a direct intrinsic combinatorial model for the 2025 maximal A_1^8 systems analogous to the Lagrangian model for E_7 .
3. Determine whether analogous unique-completion phenomena occur in other exceptional or non-crystallographic root configurations.
4. Study the incidence structures obtained from the maximal orthogonal subsystems and their intersections, including their association schemes and possible connections with coding theory.

20 Conclusion

The exceptional systems E_6 , E_7 , and E_8 exhibit markedly different global orthogonal-frame geometries.

For E_6 , maximal orthogonal systems have size four and the complete spectrum is obtained by direct orthogonal-complement enumeration.

For E_7 , reduction modulo 2 produces a canonical six-dimensional symplectic space. The 63 root lines become its nonzero vectors, and maximal orthogonal systems become Lagrangian subspaces. This gives exactly 135 maximal A_1^7 systems and proves unique completion beginning at frame size four.

For E_8 , every root complement is E_7 . This produces exactly 2025 maximal A_1^8 systems. Their intersection is bounded by four root lines, and hence frame size five is the threshold for unique completion.

The resulting high-order formulas are

$$N_k(E_7) = 135 \cdot 2^k \cdot 7^{\underline{k}} \quad (4 \leq k \leq 7),$$

and

$$N_k(E_8) = 2025 \cdot 2^k \cdot 8^{\underline{k}} \quad (5 \leq k \leq 8).$$

For

$$k \in \{5, 6, 7, 8\},$$

these agree exactly with the root-slicing identity

$$N_k(E_8) = 240 N_{k-1}(E_7).$$

Thus the enumeration of high-order orthogonal frames is controlled not merely by successive branching of root systems, but by a rigid finite-geometric completion structure.

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