Automorphic Structures Associated with the Leech Minimal Shell: Scalar Theta Obstructions, Hecke Realization, and Arithmetic Transfer Boundaries

SRFP311T1 Collaboration

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Abstract

The 196,560 minimal vectors of the Leech lattice Λ_{24} form a universally optimal spherical 11-design on $S^{23}(\sqrt{32})$. In the preceding papers of this series, the collective Riemannian Hessian on

$$\mathcal{M} = (S^{23})^{196560}, \qquad \dim \mathcal{T} = 4{,}520{,}880,$$

was reduced by Co_0-equivariance to a twelve-dimensional rational commutant. The preceding work establishes the exact rational spectrum, including

$$\lambda_{\text{ground}} = \frac{73073}{58982400}, \qquad \kappa = \frac{730}{3597}, \qquad \mathrm{tr}(H) = \frac{22608148563}{819200}.$$

The purpose of the present paper is to identify the precise boundary between this finite-group spectral structure and a possible automorphic realization. We begin with the central involution $-I \in \mathrm{Co}_0$, which gives a canonical parity decomposition of the tangent representation. We then prove an exact obstruction for the ordinary scalar harmonic theta construction: central symmetry of the Leech lattice forces every odd-degree scalar harmonic theta series to vanish identically. Thus the standard scalar theta map has a large canonical kernel and cannot, by itself, encode all twelve tangent sectors.

This obstruction leads naturally to a coefficient-system formulation. We define the vector-valued Leech theta realization problem, requiring Co_0-equivariance, nonvanishing, rational Fourier coefficients, and compatibility with a Hecke action.

We next formulate the Spectral Hecke-Realization Problem. The rational commutant

$$\mathrm{End}_{\mathrm{Co}_0}(\mathcal{T}_{\mathbb{Q}}) \cong \mathbb{Q}^{12}$$

is a finite-group statement. Multiplicity-freeness, rational character fields, and Schur index one do not by themselves produce an automorphic Hecke algebra. A genuine realization requires an explicit automorphic space, an explicit transform, and an explicitly computed Hecke action.

Finally, we distinguish Borcherds-product coefficients from prime-distribution data. The coefficients of $1/\Delta$ are colored-partition coefficients with Rademacher growth and are not the von Mangoldt function. Prime powers enter only after an Euler product or automorphic L-function has actually been constructed. We record the corresponding conditional logarithmic-derivative identity and formulate the resulting Automorphic Transfer Trace Problem.

The resulting program therefore has three logically distinct layers: exact finite spectral geometry, open automorphic realization, and open arithmetic transfer.

Contents

1 Introduction and Program Context

Let

$$X = \Lambda_{24}(4)$$

denote the minimal shell of the Leech lattice. Thus

$$|X| = 196{,}560, \qquad \|x\|^2 = 32 \quad (x \in X).$$

The configuration is a sharp spherical 11-design and is universally optimal for completely monotone potentials [3, 2].

The finite-dimensional spectral geometry of this system was established in the preceding papers of the SRFP311T1 collaboration.

1. **Equivariant Operator Compression [6].** The tangent displacement bundle was identified with the induced module
$$\mathcal{T} \cong \mathrm{Ind}_{\mathrm{Co}_2}^{\mathrm{Co}_0}(V_{\mathrm{nat}}),$$
where
$$V_{\mathrm{nat}} = u^{\perp} \subset \mathbb{R}^{24}, \qquad \dim V_{\mathrm{nat}} = 23.$$
Consequently,
$$\mathcal{T} = \bigoplus_{x \in X} T_x(S^{23}), \qquad \dim \mathcal{T} = 23 \cdot 196{,}560 = 4{,}520{,}880.$$
The collective Hessian H commutes with Co_0, reducing its determination to rational intertwining data.

2. **Collective Dynamics on $(S^{23})^{196560}$ [7].** The preceding analysis resolved the ambient Euclidean instability issue by including the second fundamental form. In particular,
$$\lambda_S = \lambda_{\perp} - c_f = \frac{1200199}{196608000} > 0,$$
and the exact 276-dimensional rotational nullspace arising from $\mathfrak{so}(24)$ was identified. The mean-curvature conservation law
$$\mathrm{tr}_{\mathcal{T}}(\mathcal{K}_f) \equiv 0$$
then gives
$$\mathrm{tr}(H) = \dim(\mathcal{T})\lambda_S = \frac{22608148563}{819200}.$$

3. **Exact Commutant Reduction and Complete Spectrum [8].** The twelve-dimensional subconstituent intertwiner space was explicitly constructed and saturated. The resulting characteristic polynomial splits over $\mathbb{Q}$ into twelve linear factors, giving the exact rational spectrum and, in particular,
$$\lambda_{\mathrm{ground}} = \frac{73073}{58982400}, \qquad \kappa = \frac{\lambda_{\mathrm{ground}}}{\lambda_S} = \frac{730}{3597}.$$

1.1 The Purpose of the Present Paper

The natural next question is not whether a twelve-dimensional finite algebra exists: that question has already been settled. The question is whether this finite spectral object admits a mathematically controlled automorphic realization.

Three logical levels must be separated:

1. **Finite-group spectral structure.** The twelve-sector decomposition and rational spectral algebra belong to finite-group representation theory and exact linear algebra.
2. **Automorphic realization.** A theta lift, modular form, automorphic representation, or Hecke module carrying the twelve sectors has not been constructed here.
3. **Arithmetic transfer.** Even after an automorphic realization exists, the passage from its Hecke data to Euler products and prime-power traces requires a further construction.

The principal exact result of the present paper is therefore a *negative structural statement*: the ordinary scalar harmonic theta map necessarily annihilates all odd-degree polynomial data.

2 The Twelve-Sector Tangent Representation

Let

$$G = \mathrm{Co}_0 = 2 \cdot \mathrm{Co}_1$$

and let $-I \in G$ denote the central inversion. Fix $u \in X$. Its stabilizer is

$$H_u = \mathrm{Stab}_G(u) \cong \mathrm{Co}_2.$$

The tangent representation at u is

$$V_{\mathrm{nat}} = T_u S^{23} = u^{\perp} \subset \mathbb{R}^{24},$$

of dimension 23.

The harmonic constituents relevant to the minimal-shell permutation representation occur in degrees

$$k = 0, 1, \ldots, 6.$$

Their dimensions are

$$1, \quad 24, \quad 299, \quad 2576, \quad 17250, \quad 95680, \quad 80730.$$

The tangent representation is

$$\mathcal{T} \cong \mathrm{Ind}_{H_u}^{G}(V_{\mathrm{nat}}),$$

and the preceding character calculation gives a multiplicity-free decomposition

$$\mathcal{T}_{\mathbb{C}} \cong \bigoplus_{j=1}^{12} V_j.$$

2.1 Central Parity Splitting

The central involution acts on the shell by

$$x \longmapsto -x.$$

Since no minimal vector is fixed by $-I$,

$$\chi_M(-I) = 0$$

for the permutation representation M on X.

On each tangent space the differential of $-I$ is again multiplication by -1. Hence

$$\chi_{\mathcal{T}}(-I) = 0.$$

Theorem 2.1 (Parity Splitting)**.** *The tangent representation admits a canonical decomposition*

$$\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-,$$

where $\mathcal{T}_\pm$ are the ± 1 eigenspaces of the central involution. Moreover,

$$\dim \mathcal{T}_+ = \dim \mathcal{T}_- = 2{,}260{,}440.$$

Proof. The action of $-I$ on $\mathcal{T}$ is an involution. Therefore its eigenvalues are ± 1. If their multiplicities are m_+ and m_-, then

$$m_+ + m_- = \dim \mathcal{T} = 4{,}520{,}880$$

while

$$m_+ - m_- = \chi_{\mathcal{T}}(-I) = 0.$$

Hence

$$m_+ = m_- = 2{,}260{,}440.$$

□

The twelve sectors are listed in Table 1.

Sector	**Description**	**Harmonic origin**	**Parity**	**Dimension**	**Eigenvalue**	**GAP ID**
V_1	$\mathfrak{so}(24)$	$k = 1$	$+1$	276	0	$X.2$
V_2	Sym^2_0	$k = 1$	$+1$	299	$\frac{797071}{737280}$	$X.3$
V_3	Hexadecapole lower mode	$k = 3$	$+1$	17,250	$\frac{872241}{10240000}$	$X.6$
V_4	Acoustic sub-band	$k = 3$	$+1$	44,275	$\frac{219791}{92160000}$	$X.9$
V_5	Higher optical mode	$k = 5$	$+1$	376,740	$\frac{199381}{18432000}$	$X.14$
V_6	Intermediate acoustic mode	$k = 5$	$+1$	1,821,600	$\frac{24731}{5760000}$	$X.24$
Total even dimension				**2,260,440**		
V_7	Dipole vector	$k = 0$	-1	24	$\frac{24913889}{6553600}$	$X.102$
V_8	Octupole lower mode	$k = 2$	-1	2,576	$\frac{432845153}{1474560000}$	$X.104$
V_9	Transverse ground state	$k = 2$	-1	4,576	$\frac{73073}{58982400}$	$X.105$
V_{10}	Octupole upper mode	$k = 4$	-1	95,680	$\frac{40598593}{1474560000}$	$X.107$
V_{11}	Odd phonon mode	$k = 4$	-1	315,744	$\frac{558817}{163840000}$	$X.110$
V_{12}	Lower optical mode	$k = 6$	-1	1,841,840	$\frac{1479317}{294912000}$	$X.114$
Total odd dimension				**2,260,440**		
Total tangent dimension				**4,520,880**		

Table 1: The twelve irreducible tangent sectors of $\mathcal{T}$.

3 The Rational Spectral Algebra

The multiplicity-free complex decomposition gives

$$\mathrm{End}_G(\mathcal{T}_{\mathbb{C}}) \cong \mathbb{C}^{12}.$$

The rational statement requires additional arithmetic information about the irreducible characters.

Theorem 3.1 (Rational Commutant [8])**.** *For every active irreducible constituent χ_j occurring in $\mathcal{T}_{\mathbb{C}}$, the character field is $\mathbb{Q}$ and the Schur index over $\mathbb{Q}$ is one. Consequently,*

$$\mathrm{End}_G(\mathcal{T}_{\mathbb{Q}}) \cong \mathbb{Q}^{12}.$$

Proof. Multiplicity-freeness gives

$$\mathrm{End}_G(\mathcal{T}_{\mathbb{C}}) \cong \mathbb{C}^{12}.$$

The character-field and Schur-index calculations establish that each active complex constituent has a rational realization with Schur index one. Therefore each simple complex factor descends to a copy of $\mathbb{Q}$. The rational commutant is consequently

$$\mathbb{Q}^{12}.$$

□

Thus there are twelve primitive rational idempotents

$$P_1, \ldots, P_{12},$$

satisfying

$$P_i P_j = \delta_{ij} P_i, \qquad \sum_{j=1}^{12} P_j = I_{\mathcal{T}}.$$

The Hessian has the spectral resolution

$$H = \sum_{j=1}^{12} \lambda_j P_j, \qquad \lambda_j \in \mathbb{Q}.$$

This finite algebra is the object for which an automorphic realization is sought.

4 The Scalar Harmonic Theta Map

Let

$$\mathrm{Harm}_k(\mathbb{R}^{24})$$

denote the space of homogeneous harmonic polynomials of degree k. For

$$P \in \mathrm{Harm}_k(\mathbb{R}^{24}),$$

define

$$\Theta_{\Lambda_{24},P}(\tau) = \sum_{x \in \Lambda_{24}} P(x) q^{\|x\|^2/2}, \qquad q = e^{2\pi i \tau}.$$

For an even unimodular lattice of rank 24, the classical harmonic theta construction gives a modular form of weight

$$12 + k$$

for $\mathrm{SL}_2(\mathbb{Z})$, subject to the standard harmonic-theta hypotheses [10].

The construction nevertheless has a canonical kernel.

4.1 Odd-Degree Vanishing

Theorem 4.1 (Odd Scalar Harmonic Theta Vanishing)**.** *For every odd integer k and every*

$$P \in \mathrm{Harm}_k(\mathbb{R}^{24}),$$

one has

$$\boxed{\Theta_{\Lambda_{24},P}(\tau) \equiv 0.}$$

Proof. The Leech lattice is centrally symmetric:

$$x \in \Lambda_{24} \quad \Longleftrightarrow \quad -x \in \Lambda_{24}.$$

If P is homogeneous of odd degree, then

$$P(-x) = -P(x).$$

Consequently, the terms indexed by x and $-x$ cancel:

$$P(x)q^{\|x\|^2/2} + P(-x)q^{\|-x\|^2/2} = 0.$$

Summing over the lattice gives

$$\Theta_{\Lambda_{24},P}(\tau) = 0.$$

□

Corollary 4.2 (Kernel of the Scalar Theta Transform)**.** *For odd k, the scalar harmonic theta transform*

$$\Theta_{\Lambda_{24},\bullet} : \mathrm{Harm}_k(\mathbb{R}^{24}) \longrightarrow M_{12+k}(\mathrm{SL}_2(\mathbb{Z}))$$

is the zero map.

Remark 4.3. The vanishing theorem is purely structural. It does not depend on a numerical truncation, a particular basis of harmonic polynomials, or a computer calculation. It follows solely from the central symmetry of Λ_{24}.

4.2 Automorphic Consequence

The result rules out one particular realization mechanism:

$$\boxed{\text{ordinary scalar harmonic theta series cannot encode odd-degree data.}}$$

It does *not* rule out automorphic realizations of the twelve sectors by other constructions. In particular, one may seek vector-valued theta series, Jacobi forms, Weil-representation-valued forms, representation-valued theta kernels, or more general theta lifts.

This distinction is essential: the obstruction is to the *ordinary scalar map*, not to automorphy itself.

5 The Vector-Valued Leech Theta Realization Problem

The natural response to the scalar obstruction is to retain information that is destroyed by scalar summation.

Definition 5.1 (Admissible Coefficient System)**.** For a sector V_j, an admissible coefficient system consists of

1. a finite-dimensional coefficient space W_j;
2. a representation
$$\rho_j : \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}(W_j);$$
3. a Co_0-equivariant coefficient map
$$\Phi_j : \Lambda_{24} \longrightarrow W_j;$$

4. a vector-valued theta series

$$\mathbf{\Theta}_{\Phi_j}(\tau) = \sum_{x\in\Lambda_{24}} \Phi_j(x) q^{\|x\|^2/2}$$

   transforming according to ρ_j.

The coefficient system separates three structures:

$$\Lambda_{24} \quad \text{(lattice)}, \qquad V_j \quad \text{(finite-group sector)}, \qquad W_j \quad \text{(automorphic coefficient space)}.$$

Problem 5.2 (Vector-Valued Leech Theta Realization)**.** For every tangent sector V_j, construct an admissible coefficient system

$$(\rho_j, W_j, \Phi_j)$$

such that:

1. $\mathbf{\Theta}_{\Phi_j}$ is nonzero;
2. the construction is Co_0-equivariant;
3. the twelve resulting automorphic objects are linearly independent;
4. their Fourier coefficients admit a rational structure;
5. the resulting automorphic space carries a compatible Hecke action.

A successful construction should also identify the precise relation between the finite-group sector V_j and the automorphic coefficient space W_j.

6 The Spectral Hecke-Realization Problem

Define the finite spectral algebra

$$\mathcal{A}_\Lambda = \mathrm{End}_{\mathrm{Co}_0}(\mathcal{T}_{\mathbb{Q}}) \cong \mathbb{Q}^{12}.$$

A Hecke realization requires more than a formal identification of two twelve-dimensional vector spaces.

Definition 6.1 (Spectral Hecke Realization)**.** A spectral Hecke realization consists of:

1. an automorphic space $\mathcal{V}$;
2. a commutative Hecke algebra
$$\mathbb{T} \subseteq \mathrm{End}(\mathcal{V});$$
3. an automorphic transform
$$T : \mathcal{T}_{\mathbb{Q}} \rightsquigarrow \mathcal{V};$$
4. an algebra homomorphism
$$\Phi : \mathcal{A}_\Lambda \longrightarrow \mathbb{T}_{\mathrm{act}} \otimes_{\mathbb{Z}} \mathbb{Q},$$
   where $\mathbb{T}_{\mathrm{act}}$ denotes the Hecke algebra acting on the automorphic image of T.

A *faithful* realization additionally requires Φ to identify the twelve primitive spectral idempotents with twelve distinct automorphic eigensystems.

Problem 6.2 (Spectral Hecke-Realization Problem)**.** Construct a faithful spectral Hecke realization of

$$\mathcal{A}_\Lambda \cong \mathbb{Q}^{12}.$$

More precisely, determine whether there exist

$$\mathcal{V}, \quad \mathbb{T}, \quad T, \quad \Phi$$

for which

$$\mathcal{A}_\Lambda \xrightarrow{\sim} \mathbb{T}_{\mathrm{act}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

and the twelve primitive idempotents

$$P_j$$

correspond to distinct automorphic eigensystems.

Proposition 6.3 (Conditional Hecke Realization of the Hessian)**.** *Assume a faithful spectral Hecke realization exists. Then the Hessian determines an element*

$$\mathcal{H} = \Phi(H) = \sum_{j=1}^{12} \lambda_j \mathbf{e}_j$$

of the active rational Hecke algebra, where $\mathbf{e}_j$ *are the corresponding automorphic idempotents. Evaluation on the associated eigensystem* π_j *gives*

$$\pi_j(\mathcal{H}) = \lambda_j.$$

Proof. By definition,

$$H = \sum_j \lambda_j P_j.$$

Applying the algebra homomorphism Φ gives

$$\Phi(H) = \sum_j \lambda_j \Phi(P_j) = \sum_j \lambda_j \mathbf{e}_j.$$

Evaluation on the j-th eigensystem annihilates the other primitive idempotents and evaluates $\mathbf{e}_j$ as 1. □

7 Comparison with Established Hecke Constructions

The research program contains a separate example in which a lattice theta construction leads to a concrete Hecke field: the rank-88, weight-44 extremal theta construction [9].

There the cuspidal projection

$$F_{43} = \Theta_{88}^{\mathrm{ext}} - E_{44}$$

belongs to

$$S_{44}(\mathrm{SL}_2(\mathbb{Z})).$$

A direct computation of a Hecke operator produces an irreducible cubic characteristic polynomial over $\mathbb{Q}$, thereby exhibiting a concrete totally real cubic Hecke field.

The methodological point for the Leech problem is not that the same Hecke field should occur, but that a genuine automorphic realization must be constructed and tested at the operator level:

$$\boxed{\text{construct } \mathcal{V} \longrightarrow \text{compute } T_n \longrightarrow \text{identify eigensystems.}}$$

No corresponding twelve-sector Hecke realization is asserted here.

8 Borcherds Products and the Partition–Prime Boundary

The arithmetic discussion requires a separate distinction.

In the Lorentzian Leech construction inside $\mathrm{II}_{25,1}$, Borcherds' product involves the coefficients of

$$\frac{1}{\Delta(\tau)} = \sum_{n=-1}^{\infty} c(n)q^n = q^{-1} + 24 + 324q + 3200q^2 + 25650q^3 + \cdots .$$

Since

$$\Delta(\tau) = q \prod_{m\geq 1} (1 - q^m)^{24},$$

the positive-index coefficients may be interpreted as 24-colored partition numbers:

$$c(n) = p_{24}(n + 1).$$

Proposition 8.1 (Partition–Prime Mismatch)**.** *The coefficients $c(n)$ of $1/\Delta$ are not the von Mangoldt function. They have support on all sufficiently large integers and exhibit Rademacher-type exponential growth, whereas*

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. The coefficients of $1/\Delta$ arise from the reciprocal Euler product

$$q^{-1} \prod_{m\geq 1} (1 - q^m)^{-24}$$

and are positive partition coefficients. Their asymptotic growth is of the form

$$c(n) \asymp n^{-27/4} e^{4\pi\sqrt{n}},$$

up to the standard Rademacher corrections.

By contrast, the von Mangoldt function is supported only at prime powers and has logarithmic size there. Hence the two sequences have fundamentally different arithmetic support and growth. □

The distinction is therefore structural rather than numerical.

9 Euler Products and Prime-Power Traces

Prime-power coefficients arise naturally from logarithmic derivatives of Euler products.

Let π be an automorphic representation with unramified local Satake parameter

$$A_p \in \mathrm{GL}_d(\mathbb{C}).$$

Its standard Euler product is formally

$$L(s, \pi) = \prod_p \det(I - A_p p^{-s})^{-1}.$$

In a region of absolute convergence,

$$-\frac{L'(s, \pi)}{L(s, \pi)} = \sum_p \sum_{k\geq 1} \frac{\mathrm{tr}(A_p^k) \log p}{p^{ks}}.$$

Equivalently,

$$-\frac{L'(s,\pi)}{L(s,\pi)} = \sum_{n\geq 1} \frac{\Lambda_\pi(n)}{n^s},$$

where

$$\Lambda_\pi(p^k) = \mathrm{tr}(A_p^k)\log p$$

and $\Lambda_\pi(n) = 0$ away from prime powers.

Proposition 9.1 (Conditional Transfer Trace Identity)**.** *Suppose a family of trace-class operators $\mathcal{L}_s$ is defined in a region of absolute convergence and satisfies*

$$\det(I - \mathcal{L}_s) = L(s,\pi)^{-1}.$$

Then

$$\boxed{-\frac{d}{ds}\log\det(I-\mathcal{L}_s) = -\frac{L'(s,\pi)}{L(s,\pi)} = \sum_{n\geq 1}\frac{\Lambda_\pi(n)}{n^s}.}$$

Proof. For a trace-class operator,

$$-\log\det(I-\mathcal{L}_s) = \sum_{r\geq 1}\frac{\mathrm{Tr}(\mathcal{L}_s^r)}{r}$$

in the region of convergence. The assumed determinant identity gives

$$-\log\det(I-\mathcal{L}_s) = \log L(s,\pi).$$

Differentiation therefore yields

$$-\frac{d}{ds}\log\det(I-\mathcal{L}_s) = -\frac{L'(s,\pi)}{L(s,\pi)}.$$

The Euler product expansion gives the prime-power expression. □

Problem 9.2 (Automorphic Transfer Trace Problem)**.** Construct an explicit dynamical family of transfer operators

$$\mathcal{L}_s$$

associated with the Leech configuration manifold

$$\mathcal{M} = (S^{23})^{196560}$$

and an explicitly constructed automorphic representation

$$\pi_{\Lambda_{24}}$$

such that, in a nonempty region of convergence,

$$\det(I-\mathcal{L}_s) = L(s,\pi_{\Lambda_{24}})^{-1}.$$

The construction should identify:

1. the underlying phase space or symbolic dynamics;
2. the transfer kernel;
3. the trace-class or nuclearity framework;
4. the relation between periodic-orbit data and local Hecke data;
5. the resulting Euler product.

10 Logical Boundaries of the Program

The results of the present paper establish several implications and several non-implications.

10.1 What the Finite Spectral Calculation Gives

The preceding work gives an exact finite algebra

$$\mathcal{A}_\Lambda \cong \mathbb{Q}^{12}$$

together with twelve rational spectral idempotents.

This is strong information about the Co_0-module $\mathcal{T}_\mathbb{Q}$ and the Hessian acting on it.

10.2 What It Does Not Give Automatically

The identity

$$\mathcal{A}_\Lambda \cong \mathbb{Q}^{12}$$

does not, by itself, imply the existence of:

1. twelve modular eigenforms;
2. twelve automorphic representations;
3. a Hecke algebra canonically isomorphic to $\mathbb{Q}^{12}$;
4. an L-function naturally attached to the Hessian;
5. a transfer operator with that L-function as Fredholm determinant.

These require additional constructions.

In particular, rational Schur index one is a statement about the rational realization of finite-group representations. A Hecke algebra is an independent arithmetic object, and an identification between the two must be exhibited rather than inferred from dimension alone.

11 Epistemic Status of the Program

Table 2 summarizes the logical status of the principal statements.

Statement / Milestone	**Tier**	**Scientific Status**
Leech tangent dimension $\dim \mathcal{T} = 4{,}520{,}880$	I	Exact theorem
Multiplicity-free twelve-sector decomposition	I	Exact character computation
Rational commutant $\mathrm{End}_{\mathrm{Co}_0}(\mathcal{T}_\mathbb{Q}) \cong \mathbb{Q}^{12}$	I	Exact finite-group result
Complete rational Hessian spectrum	I	Exact result of preceding series
$6+6$ central parity splitting	I	Exact consequence of $\chi_\mathcal{T}(-I) = 0$
Odd scalar harmonic theta vanishing	I	Exact symmetry obstruction
Direct identification $c(n) = \Lambda(n)$	I	Ruled out structurally
Euler-product logarithmic derivative	I	Exact conditional identity
Vector-valued Leech theta realization	VI	Open construction problem
Faithful Hecke realization of $\mathbb{Q}^{12}$	VI	Open problem
Explicit Leech-derived automorphic representation	VI	Open problem
Leech-derived dynamical transfer operator	VI	Open frontier

Table 2: Epistemic classification of the principal statements in this paper.

12 Conclusion

The preceding papers establish an exact twelve-sector rational spectral algebra for the tangent Hessian of the Leech minimal shell:

$$\mathrm{End}_{\mathrm{Co}_0}(\mathcal{T}_{\mathbb{Q}}) \cong \mathbb{Q}^{12}.$$

The present paper determines the first precise obstruction to a naive automorphic realization.

First, ordinary scalar harmonic theta series cannot detect odd-degree polynomial data. The obstruction is exact and follows from the identity

$$\Lambda_{24} = -\Lambda_{24}.$$

Second, this obstruction identifies the appropriate next problem rather than ending the program: one must construct a richer coefficient system capable of retaining the information annihilated by scalar summation.

Third, the existence of the finite rational algebra does not imply the existence of a Hecke algebra realization. Such a realization requires an explicit automorphic space, transform, Hecke action, and identification of the twelve spectral idempotents with automorphic eigensystems.

Fourth, the Borcherds coefficients associated with $1/\Delta$ are partition coefficients, not prime-distribution coefficients. Prime powers enter only through Euler products and their logarithmic derivatives. Any dynamical prime trace attached to the Leech spectral system therefore requires an actual automorphic L-function and a corresponding transfer construction.

The program can consequently be represented as the following sequence of mathematically distinct arrows:

$$\boxed{\begin{array}{ccccc}
\text{Leech geometry} & \longrightarrow & \mathbb{Q}^{12}\ \text{spectral algebra} & & \\
 & & \downarrow & \text{open Hecke realization} & \\
 & & \text{automorphic representation} & & \\
 & & \downarrow & \text{open transfer construction} & \\
 & & L(s,\pi_\Lambda) & \longrightarrow & -\dfrac{L'}{L}(s,\pi_\Lambda) \\
 & & & & \downarrow \\
 & & & & \text{prime-power trace.}
\end{array}}$$

The significance of the present result is therefore not a claim that the automorphic endpoint has already been reached. Rather, it identifies the first exact boundary condition that any successful construction must cross: the scalar harmonic theta map is insufficient, and the missing information must be carried by an explicitly constructed coefficient system.

A GAP Character Induction Certificate

The following listing records the character-theoretic audit used for the twelve-sector decomposition and rationality checks.

```
LoadPackage("ctbllib");
LoadPackage("wedderga");

g_tbl := CharacterTable("2.Co1");
h_tbl := CharacterTable("Co2");

# Central involution -I class
c_minus_I := First([1..NrConjugacyClasses(g_tbl)],
  i -> SizesConjugacyClasses(g_tbl)[i] = 1 and
       OrdersClassRepresentatives(g_tbl)[i] = 2);
```

```
# Natural 24-dimensional character
idx_24 := First([1..Length(Irr(g_tbl))],
  i -> Irr(g_tbl)[i][1] = 24 and
       Irr(g_tbl)[i][c_minus_I] = -24);

chi24 := Irr(g_tbl)[idx_24];

# Induced permutation character
fus := FusionConjugacyClasses(h_tbl, g_tbl);
chiM := InducedClassFunction(h_tbl, g_tbl,
                             TrivialCharacter(h_tbl));

# Tangent character
chiT := chiM * (chi24 - TrivialCharacter(g_tbl));

# Multiplicities
mults := List(Irr(g_tbl),
              chi -> ScalarProduct(g_tbl, chiT, chi));

active := Filtered([1..Length(Irr(g_tbl))],
                   i -> mults[i] <> 0);

cert_mult_free :=
  ForAll(mults, m -> m in [0,1]) and
  Length(active) = 12;

cert_rational :=
  ForAll(active, i ->
    CharacterField(g_tbl, [Irr(g_tbl)[i]]) = Rationals);

cert_schur :=
  ForAll(active, i ->
    SchurIndex(Irr(g_tbl)[i]) = 1);

Print("Multiplicity-Free (12 sectors): ",
      cert_mult_free, "\n");

Print("All Character Fields == Q     : ",
      cert_rational, "\n");

Print("All Schur Indices == 1        : ",
      cert_schur, "\n");
```

Listing 1: GAP character audit for the tangent representation.

B Lean 4 Rational Arithmetic Certificate

The following Lean artifact verifies the rational identities used for the trace, screening ratio, parity dimensions, and spectral trace sum.

```
import Mathlib.Data.Rat.Basic
import Mathlib.Data.Matrix.Basic
import Mathlib.Tactic

namespace LeechAutomorphicBoundary

def tangent_dim : Rat := 4520880

def lambda_S : Rat :=
  1200199 / 196608000

def lambda_ground : Rat :=
  73073 / 58982400

def screening_ratio : Rat :=
  lambda_ground / lambda_S

theorem screening_ratio_eval :
    screening_ratio = 730 / 3597 := by
```

```
  norm_num [screening_ratio, lambda_ground, lambda_S]

def total_trace : Rat :=
  tangent_dim * lambda_S

theorem total_trace_eval :
    total_trace = 22608148563 / 819200 := by
  norm_num [total_trace, tangent_dim, lambda_S]

def d_even : List Rat :=
  [276, 299, 17250, 44275, 376740, 1821600]

def d_odd : List Rat :=
  [24, 2576, 4576, 95680, 315744, 1841840]

theorem even_parity_dim :
    d_even.sum = 2260440 := by
  norm_num [d_even]

theorem odd_parity_dim :
    d_odd.sum = 2260440 := by
  norm_num [d_odd]

theorem total_bundle_dim :
    d_even.sum + d_odd.sum = tangent_dim := by
  norm_num [d_even, d_odd, tangent_dim]

def lambda_list : List Rat :=
  [
    0,
    797071 / 737280,
    872241 / 10240000,
    219791 / 92160000,
    199381 / 18432000,
    24731 / 5760000,
    24913889 / 6553600,
    432845153 / 1474560000,
    73073 / 58982400,
    40598593 / 1474560000,
    558817 / 163840000,
    1479317 / 294912000
  ]

def d_all : List Rat :=
  d_even ++ d_odd

theorem spectrum_trace_sum_rule :
    (List.zipWith (  *  ) d_all lambda_list).sum
      = total_trace := by
  norm_num [
    d_all,
    d_even,
    d_odd,
    lambda_list,
    total_trace,
    tangent_dim,
    lambda_S
  ]

end LeechAutomorphicBoundary
```

Listing 2: Lean 4 exact rational arithmetic certificate.

References

[1] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), 405–444.

[2] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska, *Universal optimality of the E_8 and Leech lattices and interpolation formulas*, Ann. of Math. (2) **196** (2022), no. 3, 983–1082.

[3] H. Cohn and A. Kumar, *Universally optimal distribution of points on spheres*, J. Amer. Math. Soc. **20** (2007), no. 1, 99–148.

[4] J. H. Conway, *A perfect group of order* $8,315,553,613,086,720,000$ *and the sporadic simple groups*, Proc. Natl. Acad. Sci. USA **61** (1968), no. 2, 398–400.

[5] P. Delsarte, J.-M. Goethals, and J. J. Seidel, *Spherical codes and designs*, Geom. Dedicata **6** (1977), no. 3, 363–388.

[6] SRFP311T1 Collaboration, *Multiplicity-Free Equivariant Operator Compression of the Leech Minimal Shell*, Preprint (Paper 1), 2026.

[7] SRFP311T1 Collaboration, *Exact Collective Dynamics on the Leech Minimal Shell*, Preprint (Paper 2), 2026.

[8] SRFP311T1 Collaboration, *Exact Commutant Reduction and Complete Rational Spectrum of the Leech Minimal Shell on* $(S^{23})^{196560}$, Preprint (Paper 3), 2026.

[9] SRFP311T1 Collaboration, *The Extremal Theta Series in Rank 88, the Weight-44 Hecke Field, and Quadratic-Twist Periods*, Preprint, 2026.

[10] B. Schoeneberg, *Elliptic Modular Functions*, Springer-Verlag, Berlin-Heidelberg, 1974.

[11] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer-Verlag, New York, 1999.