Exact Rational Spherical Designs on the Leech Lattice through Strength 111

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Abstract

We develop an exact rational theory of multi-shell spherical cubatures on the 24-dimensional Leech lattice Λ_{24}, bridging modular forms on $\mathrm{SL}_2(\mathbb{Z})$, convex cone duality, sporadic group representation theory under $\mathrm{Co}_0 = \mathrm{Aut}(\Lambda_{24})$, and spherical design theory. By applying Reynolds group averaging over Co_0, degree-k harmonic moments vanish identically on every shell outside the Co_0-invariant subspace $(\mathrm{Harm}_k(\mathbb{R}^{24}))^{\mathrm{Co}_0}$. By Venkov's isomorphism theorem, the weighted theta series defines an isomorphism between $(\mathrm{Harm}_k(\mathbb{R}^{24}))^{\mathrm{Co}_0}$ and the cusp space $S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})) = \Delta^2 M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$. Consequently, the infinite system of degree-k harmonic cubature equations collapses to $C(k) = \sum_{j=1}^{k/2} \dim M_{2j-12}(\mathrm{SL}_2(\mathbb{Z})) = \frac{k^2}{48} - \frac{k}{4} + \delta(k)$ scalar modular equations.

We formulate the multi-shell cubature problem as an exact rational homogeneous linear system $AW = 0$. Under the full-row-rank condition $\mathrm{rank}(A) = C$, Farkas–Gordan cone duality establishes that a strictly positive rational weight vector exists if and only if no non-trivial cusp form in $S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z}))$ has non-negative Fourier coefficients across the active window.

Using fraction-free Bareiss integer elimination and Dixon p-adic lifting, we audit the multi-shell system through degree $k = 110$ (strength $t = 111$). Every configuration is audited under a two-sided protocol: exact rational vanishing of all imposed moments ($AW = 0$ in $\mathbb{Q}^C$) and exact non-vanishing of the first non-trivial omitted cusp space ($\Phi_W \neq 0$). We resolve the previously reported failures at strengths $t \in \{21, 25, 29, 33\}$ by skipping the resonant shell S_3 (where $a_3(\Delta^2) = -48 < 0$) or shifting boundaries, certifying positive rational designs through strength 111 spanning up to 226 consecutive shells ($S_9 \ldots S_{234}$) with exact rational denominators reaching 1,580 decimal digits (5,248 binary bits).

For degrees $k \equiv 2 \pmod{12}$, an exhaustive search over all predecessor windows proves that the minimal positive starting shell is $m^{\min}_{\mathrm{start}}(k) = (k-2)/12$ for all audited cases $k \in \{62, 74, 86, 98, 110\}$, corresponding to the threshold where the discriminant filtration clears the Serre obstruction $M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$. By modular cone duality, this certifies unconditionally that every non-zero cusp form in $\Delta^2 M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$ changes sign within the discrete interval $[(k-2)/12, (k-2)/12 + C(k)]$.

Finally, decomposing Shell 8 ($m = 8$, norm 16) under Co_0 into the doubled minimal orbit $\mathcal{O}_{8_A} = 2 \cdot S_2$ and primitive octad sub-orbits, we discover an exact rational harmonic sign inversion under the canonical invariant Ψ_{12}: $\Psi_{12}(\mathcal{O}_{8_A})/\Psi_{12}(S_2) = +4{,}096$ versus $\Psi_{12}(v_B)/\Psi_{12}(S_2) = -40/23$, demonstrating that monolithic shells blur opposing harmonic moments. On minimal spherical supports, $S_2 \cup S_3$ forms an exact spherical 15-design on 16,969,680 distinct points with closed-form rational weights $(W_2, W_3) = (1, 243/1024)$. On the 12-dimensional intertwiner space $\mathcal{M} = \mathrm{Hom}_K(V_{\mathrm{nat}}, T_{S_2}S^{23})$ under $K \cong 2 \cdot \mathrm{Co}_2$, we formulate the local Hessian problem for completely monotonic potentials via Bernstein–Widder theory.

Contents

1 Introduction

A weighted spherical t-design on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is a finite set of points X together with positive weights $W : X \to \mathbb{R}_{>0}$ that integrates every polynomial of degree at most t exactly against the normalized Haar measure σ on S^{d-1}:

$$\int_{S^{d-1}} f(\xi)\, d\sigma(\xi) = \frac{\sum_{x\in X} W(x) f(x)}{\sum_{x\in X} W(x)}, \qquad \forall f \in \mathbb{R}[x_1, \ldots, x_d]_{\leq t}. \tag{1.1}$$

Equivalently, X is a spherical t-design if and only if for every non-constant homogeneous harmonic polynomial $P_k \in \mathrm{Harm}_k(\mathbb{R}^d)$ of degree $1 \leq k \leq t$, the weighted cubature moment vanishes:

$$\sum_{x\in X} W(x) P_k(x) = 0, \qquad \forall P_k \in \mathrm{Harm}_k(\mathbb{R}^d), \qquad 1 \leq k \leq t. \tag{1.2}$$

For *equal-weight* spherical designs ($W(x) \equiv 1$), the classical Delsarte–Goethals–Seidel (DGS) bound [5] establishes that for any odd strength $t = 2e + 1$ in dimension d:

$$N_{\mathrm{DGS}}(t) \geq 2\binom{d+e-1}{e} = 2\binom{d+\frac{t-1}{2}-1}{\frac{t-1}{2}}. \tag{1.3}$$

In dimension $d = 24$ (the unit sphere S^{23}), this lower bound grows asymptotically as:

$$N_{\mathrm{DGS}}(t) \geq 2\binom{23+\frac{t-1}{2}}{\frac{t-1}{2}} = \Theta(t^{23}) \qquad (\text{as } t \to \infty). \tag{1.4}$$

For general *positive-weight* cubatures, the fundamental dimension-counting lower bound takes the form:

$$N_{\mathrm{pos}}(t) \geq \dim \mathcal{P}_{\lfloor t/2 \rfloor}(S^{d-1}) = \binom{d+\lfloor t/2\rfloor-1}{d-1} + \binom{d+\lfloor t/2\rfloor-2}{d-1} = \Theta(t^{d-1}), \tag{1.5}$$

which is also of order $\Theta(t^{23})$ in dimension $d = 24$.

The Leech lattice $\Lambda_{24} \subset \mathbb{R}^{24}$ is the unique even unimodular lattice without roots [4]. We define the shell S_m as the set of vectors of squared norm $2m$:

$$S_m = \{x \in \Lambda_{24} : \|x\|^2 = 2m\}, \qquad m \geq 2. \tag{1.6}$$

Venkov [7] proved that every individual shell S_m, radially projected to the unit sphere via $x \mapsto x/\sqrt{2m}$, forms an exact spherical 11-design. In particular, for the minimal shell S_2 ($\|x\|^2 = 4$), $|S_2| = 196{,}560$, which matches Delsarte's bound $2\binom{28}{5} = 196{,}560$ identically, making S_2 a tight spherical 11-design. Furthermore, because $\dim M_2(\mathrm{SL}_2(\mathbb{Z})) = 0$, all degree-14 harmonic moments vanish identically on every shell individually.

In 2013, Bondarenko, Radchenko, and Viazovska [2] proved the Korevaar–Meyers conjecture, establishing the existence of spherical t-designs on S^{d-1} with $\mathcal{O}(t^d)$ points for all t and all d. However, their proof was non-constructive, relying on topological degree theory. A fundamental open problem has been whether an explicit, exact rational family of high-strength spherical designs can be constructed directly from a Euclidean lattice.

In this paper, we construct exact rational multi-shell cubatures on Λ_{24} through degree $k = 110$ (strength $t = 111$). We resolve the intermediate gaps via resonant shell skipping, prove that boundary shifts follow an exact mod-12 quantization law governed by convex cone duality and the Serre obstruction $M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$, establish deterministic sign changes of cusp forms up to weight 122, and demonstrate that sub-orbit splitting yields compressed designs.

2 Leech Shells and Harmonic Theta Series

2.1 Leech Shell Cardinalities

The theta series of the Leech lattice is a modular form of weight 12 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$:

$$\Theta_{\Lambda_{24}}(\tau) = \sum_{x\in\Lambda_{24}} q^{\|x\|^2/2}, \qquad q = e^{2\pi i\tau}, \quad \tau \in \mathbb{H}. \tag{2.1}$$

Since Λ_{24} has no roots (no vectors of squared norm 2), the q^1 term vanishes. Spanning $M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ with the Eisenstein series E_{12} and the Ramanujan discriminant Δ, one obtains:

$$\Theta_{\Lambda_{24}}(\tau) = E_{12}(\tau) - \frac{65520}{691}\Delta(\tau) = 1 + \sum_{m=2}^{\infty} N_m q^m. \tag{2.2}$$

Comparing Fourier coefficients yields the exact shell cardinalities:

$$N_m = |S_m| = \frac{65520}{691}\big(\sigma_{11}(m) - \tau(m)\big), \qquad m \geq 2, \tag{2.3}$$

where $\sigma_{11}(m) = \sum_{d|m} d^{11}$ and $\tau(m)$ is Ramanujan's tau function. By Ramanujan's congruence $\sigma_{11}(m) \equiv \tau(m) \pmod{691}$, N_m is an exact integer for all m.

Table 2.1: Exact arithmetic of the first eight nonzero Leech shells.

m	$\sigma_{11}(m)$	$\tau(m)$	$\sigma_{11}(m)-\tau(m)$	$\dfrac{\sigma_{11}(m)-\tau(m)}{691}$	N_m
2	2 049	−24	2 073	3	196 560
3	177 148	252	176 896	256	16 773 120
4	4 196 353	−1 472	4 197 825	6 075	398 034 000
5	48 828 126	4 830	48 823 296	70 656	4 629 381 120
6	362 976 252	−6 048	362 982 300	525 300	34 417 656 000
7	1 977 326 744	−16 744	1 977 343 488	2 861 568	187 489 935 360
8	8 594 130 945	84 480	8 594 046 465	12 437 115	814 879 774 800
9	31 381 236 757	−113 643	31 381 350 400	45 414 400	2 975 551 488 000

2.2 Harmonic Theta Series

Let $P_k \in \mathrm{Harm}_k(\mathbb{R}^{24})$ be a homogeneous harmonic polynomial of degree k. The weighted lattice theta series:

$$\Theta_{\Lambda_{24},P_k}(\tau) = \sum_{x\in\Lambda_{24}} P_k(x) q^{\|x\|^2/2} = \sum_{m=2}^{\infty}\left(\sum_{x\in S_m} P_k(x)\right) q^m \tag{2.4}$$

is a cusp form of weight $k+12$ on $\mathrm{SL}_2(\mathbb{Z})$. Because Λ_{24} has no vectors of norm 2, the Fourier series vanishes to order at least 2 at the cusp $q = 0$. Thus:

$$\Theta_{\Lambda_{24},P_k} \in S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})) := \{f \in S_{k+12}(\mathrm{SL}_2(\mathbb{Z})) : \mathrm{ord}_{q=0}(f) \geq 2\}. \tag{2.5}$$

2.3 Reynolds Projection and Shell Annihilation

Let $G = \mathrm{Co}_0 = \mathrm{Aut}(\Lambda_{24}) \subset O(24)$. For any $P \in \mathrm{Harm}_k(\mathbb{R}^{24})$, its Reynolds group-averaging projection onto the G-invariant subspace $(\mathrm{Harm}_k(\mathbb{R}^{24}))^G$ is defined by:

$$\overline{P}(x) = \frac{1}{|G|}\sum_{g\in G} P(gx). \tag{2.6}$$

Proposition 2.1 (Shell Annihilation Outside the Invariant Subspace)**.** *If* $P_k \in \mathrm{Harm}_k(\mathbb{R}^{24})$ *is orthogonal to* $(\mathrm{Harm}_k(\mathbb{R}^{24}))^{\mathrm{Co}_0}$ *under the standard* $L^2(S^{23})$ *inner product, then its shell sum vanishes identically:*

$$\sum_{x\in S_m} P_k(x) = 0 \qquad \forall m \geq 2. \tag{2.7}$$

Proof. Because the automorphism group $G = \mathrm{Co}_0$ preserves every lattice shell S_m setwise, re-indexing the summation gives:

$$\sum_{x\in S_m} P_k(x) = \frac{1}{|G|}\sum_{g\in G}\sum_{x\in S_m} P_k(gx) = \sum_{x\in S_m}\left(\frac{1}{|G|}\sum_{g\in G} P_k(gx)\right) = \sum_{x\in S_m} \overline{P}_k(x). \tag{2.8}$$

The Reynolds operator is the orthogonal projection onto $(\mathrm{Harm}_k(\mathbb{R}^{24}))^G$. Since $P_k \perp (\mathrm{Harm}_k(\mathbb{R}^{24}))^G$, $\overline{P}_k \equiv 0$, establishing that the shell moment vanishes on every shell individually. □

Consequently, only Co_0-invariant harmonics can impose non-trivial constraints across multi-shell assemblies.

2.4 The Cusp Space Quotient and Venkov Isomorphism

The modular discriminant:

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 - \cdots \in S_{12}(\mathrm{SL}_2(\mathbb{Z})) \tag{2.9}$$

has a simple zero at the cusp and no zeros in the upper half-plane $\mathbb{H}$.

Lemma 2.2 (Cusp Space Quotient Isomorphism)**.** *For even* $k \geq 12$*, division by* Δ^2 *defines a linear isomorphism:*

$$S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})) \xrightarrow{\sim} M_{k-12}(\mathrm{SL}_2(\mathbb{Z})), \qquad f(\tau) \longmapsto \frac{f(\tau)}{\Delta(\tau)^2}. \tag{2.10}$$

Consequently, $\dim S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})) = \dim M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$.

Proof. If $f \in S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z}))$, f vanishes to order at least 2 at the cusp. Since Δ^2 has exactly a double zero at ∞ and no zeros in $\mathbb{H}$, the quotient f/Δ^2 is holomorphic on $\mathbb{H}$ and at ∞, transforming with modular weight $(k+12)-24 = k-12$. The inverse map is multiplication by Δ^2. □

Theorem 2.3 (Venkov's Harmonic Theta Isomorphism Theorem [7])**.** *For every even degree* $k \geq 12$*, the weighted theta series mapping:*

$$\Theta : (\mathrm{Harm}_k(\mathbb{R}^{24}))^{\mathrm{Co}_0} \longrightarrow S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})), \qquad P \longmapsto \Theta_{\Lambda_{24},P}(\tau) \tag{2.11}$$

is an isomorphism of vector spaces. Consequently, $\dim(\mathrm{Harm}_k(\mathbb{R}^{24}))^{\mathrm{Co}_0} = \dim M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$.

Proof. By Venkov's Theorem [7, Theorem 3.1], for any even unimodular lattice without roots in dimension 24, the mapping Θ is an isomorphism onto the subspace of cusp forms of weight $k+12$ vanishing to order at least 2 at ∞, which is precisely $S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z}))$. Combining this with Lemma 2.2 completes the proof. □

Thus, the scalar modular equations indexed by a basis of $M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$ provide necessary and sufficient conditions for spherical cubature. We choose the monomial basis:

$$F_{k,b} = \Delta^2 E_4^{a_b} E_6^{b_b}, \qquad 4a_b + 6b_b = k-12, \quad a_b, b_b \in \mathbb{Z}_{\geq 0}. \tag{2.12}$$

3 Dimension Counts and Cumulative Condition Growth

For even $k \geq 12$, the cumulative number of even-degree modular conditions is:

$$C(k) = \sum_{j=1}^{k/2} \dim M_{2j-12}(\mathrm{SL}_2(\mathbb{Z})). \tag{3.1}$$

For even weights $w \geq 0$, the dimension formula gives:

$$\dim M_w(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} 0, & w < 0 \text{ or } w = 2, \\ \left\lfloor \dfrac{w}{12} \right\rfloor, & w \equiv 2 \pmod{12}, \\ \left\lfloor \dfrac{w}{12} \right\rfloor + 1, & w \not\equiv 2 \pmod{12}. \end{cases} \tag{3.2}$$

Proposition 3.1 (Exact Asymptotic Formula for $C(k)$)**.** *For even $k \geq 12$, the cumulative conditions satisfy:*

$$C(k) = \frac{k^2}{48} - \frac{k}{4} + \delta(k), \tag{3.3}$$

where $\delta(k)$ is a periodic function of period 12 taking values in $\{5/12, 2/3, 3/4, 1\}$.

Proof. Let $m = k/2$. The weights $w_j = 2(j-6)$ for $j < 6$ satisfy $\dim M_{w_j} = 0$. Setting $i = j - 6$, $C(k) = \sum_{i=0}^{N} \dim M_{2i}(\mathrm{SL}_2(\mathbb{Z}))$ where $N = m - 6 = k/2 - 6$. For $i \geq 0$, $\dim M_{2i}(\mathrm{SL}_2(\mathbb{Z})) = \lfloor i/6 \rfloor + \epsilon(i)$, where $\epsilon(i) = 0$ if $i \equiv 1 \pmod 6$ and $\epsilon(i) = 1$ otherwise (with $\dim M_2 = 0$ matching $\epsilon(1) = 0$).

Decompose $C(k) = S_1 + S_2$ with $S_1 = \sum_{i=0}^{N} \lfloor i/6 \rfloor$ and $S_2 = \sum_{i=0}^{N} \epsilon(i)$. Write $N = 6q + r$ with $q = \lfloor N/6 \rfloor$ and $r \in \{0, 1, 2, 3, 4, 5\}$. Summing over complete blocks of 6 gives:

$$S_1 = 6 \sum_{p=0}^{q-1} p + (r+1)q = 3q^2 + (r-2)q = \frac{N^2 - 4N - r^2 + 4r}{12}, \tag{3.4}$$

$$S_2 = 5q + \sum_{j=0}^{r} \epsilon(j) = \frac{5N}{6} + \left(\sum_{j=0}^{r} \epsilon(j) - \frac{5r}{6} \right). \tag{3.5}$$

Adding S_1 and S_2 yields:

$$C(k) = \frac{N^2 + 6N}{12} + \Delta(r), \qquad \Delta(r) = \frac{-r^2 - 6r}{12} + \sum_{j=0}^{r} \epsilon(j). \tag{3.6}$$

Substituting $N = k/2 - 6$ into the quadratic term:

$$\frac{N^2 + 6N}{12} = \frac{(k/2 - 6)^2 + 6(k/2 - 6)}{12} = \frac{\frac{k^2}{4} - 3k}{12} = \frac{k^2}{48} - \frac{k}{4}. \tag{3.7}$$

The remainder $\delta(k) = \Delta(r)$ depends only on $r = (k/2 - 6) \pmod 6$:

- $r = 0$ ($k \equiv 0 \pmod{12}$): $\Delta(0) = 1$.
- $r = 1$ ($k \equiv 2 \pmod{12}$): $\Delta(1) = -7/12 + 1 = 5/12$.
- $r = 2$ ($k \equiv 4 \pmod{12}$): $\Delta(2) = -16/12 + 2 = 2/3$.
- $r = 3$ ($k \equiv 6 \pmod{12}$): $\Delta(3) = -27/12 + 3 = 3/4$.
- $r = 4$ ($k \equiv 8 \pmod{12}$): $\Delta(4) = -40/12 + 4 = 2/3$.
- $r = 5$ ($k \equiv 10 \pmod{12}$): $\Delta(5) = -55/12 + 5 = 5/12$.

This completes the exact proof for all even $k \geq 12$. □

4 The Exact Rational Cubature Protocol

4.1 Moment Equations

Let $\{F_1, \ldots, F_C\}$ be the basis of $S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z}))$ defined in Eq. (2.12). For an active window of $M = C + 1$ shells $\{S_{m_1}, \ldots, S_{m_M}\}$, assigning a scalar weight W_{m_j} to each vector in shell S_{m_j} scales degree-k harmonics by $(2m_j)^{-k/2}$ upon radial projection. The moment conditions are:

$$\sum_{j=1}^{M} W_{m_j}(2m_j)^{-k/2} a_{m_j}(F_b) = 0, \qquad 1 \le b \le C. \tag{4.1}$$

Defining $A_{b,j} = (2m_j)^{-k/2} a_{m_j}(F_b) \in \mathbb{Q}$, the system becomes $AW = 0$ with $A \in \mathbb{Q}^{C\times M}$. When $\mathrm{rank}(A) = C$, normalizing $W_{m_1} = 1$ yields the inhomogeneous system:

$$M_1 \widetilde{W} = -A_{*,1}, \qquad M_1 \in \mathbb{Q}^{C\times C}, \qquad \widetilde{W} = (W_{m_2}, \ldots, W_{m_{C+1}})^T. \tag{4.2}$$

4.2 Two-Sided Certification of Exact Strength

Because $\dim M_2(\mathrm{SL}_2(\mathbb{Z})) = 0$, all degree-14 moments vanish identically on every shell. Thus, exact strength certification requires auditing the first omitted degree where non-trivial cusp forms exist.

Definition 4.1 (Two-Sided Exact-Strength Certification)**.** Let $k_{\max}$ be the maximum degree of imposed equations. Define k_{next} as the smallest even degree $k > k_{\max}$ for which $\dim M_{k-12}(\mathrm{SL}_2(\mathbb{Z})) > 0$. A positive rational weight vector $W \in \mathbb{Q}^M_{>0}$ is certified at exact strength $t = k_{\mathrm{next}} - 1$ if and only if:

1. **Exact imposed vanishing:** $AW = 0$ identically in $\mathbb{Q}^C$.

2. **First omitted non-vanishing:** The cubature functional evaluated on the first omitted cusp space is strictly non-zero:

$$\exists F \in S^0_{k_{\mathrm{next}}+12}(\mathrm{SL}_2(\mathbb{Z})) \quad \text{such that} \quad L_W(F) = \sum_{j=1}^{M} W_{m_j}(2m_j)^{-k_{\mathrm{next}}/2} a_{m_j}(F) \neq 0. \tag{4.3}$$

In particular, for $k_{\max} = 12$, $k_{\mathrm{next}} = 16$ (since $\dim M_2 = 0$), so cancellation of degree 12 certifies exact strength $t = 15$.

5 Farkas and Gordan Cone Duality

5.1 The Modular Moment Cone

Fix an active shell window $\{s, \ldots, s+C\}$. The column vectors $v(m) = \big((2m)^{-k/2} a_m(F_1), \ldots, (2m)^{-k/2} a_m(F_C)\big)^T \in$ $\mathbb{R}^C$ generate the convex cone:

$$\mathcal{C}_k(s) = \mathrm{cone}\{v(s), v(s+1), \ldots, v(s+C)\} \subset \mathbb{R}^C. \tag{5.1}$$

Theorem 5.1 (Modular Gordan–Farkas Alternative)**.** *Suppose the moment matrix $A \in \mathbb{Q}^{C\times(C+1)}$ has full row rank $\mathrm{rank}(A) = C$. Then the following conditions are equivalent:*

1. *There exists a strictly positive rational cubature weight vector $W \in \mathbb{Q}^{C+1}_{>0}$ such that $AW = 0$.*

2. *There is no non-zero functional $c \in \mathbb{R}^C \setminus \{0\}$ such that $A^T c \ge 0$ coordinatewise.*

3. *There is no non-zero cusp form* $F_c = \sum_{i=1}^{C} c_i F_i \in S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})) \setminus \{0\}$ *whose Fourier coefficients are non-negative across the entire window:*

$$\nexists F \in S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})) \setminus \{0\} \quad \text{such that} \quad a_m(F) \geq 0 \quad \forall m \in \{s, s+1, \ldots, s+C\}. \tag{5.2}$$

Proof. The equivalence of (1) and (2) is Gordan's theorem of the alternative applied to the kernel of A. For (2) $\iff$ (3), note that $(A^T c)_j = c^T v(m_j) = (2m_j)^{-k/2} a_{m_j}(F_c)$. Since $(2m_j)^{-k/2} > 0$, $A^T c \geq 0$ is strictly equivalent to $a_{m_j}(F_c) \geq 0$ for all active shells.

If $W \in \mathbb{Q}^{C+1}_{>0}$ satisfies $AW = 0$, then for any c with $A^T c \geq 0$, we have $0 = c^T(AW) = (A^T c)^T W$. Because $W_j > 0$ and $(A^T c)_j \geq 0$, every component must vanish: $(A^T c)_j = 0$ for all j. Since $\mathrm{rank}(A) = C$, this implies $c = 0$, completing the equivalence. □

Corollary 5.2 (Finite-Window Deterministic Sign Change)**.** *Under the hypotheses of Theorem 5.1, if an exact positive cubature weight vector* $W > 0$ *exists, then every non-zero cusp form* $F \in S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})) \setminus \{0\}$ *must have at least one strictly positive and at least one strictly negative Fourier coefficient within the window:*

$$\min_{m \in [s, s+C]} a_m(F) < 0 < \max_{m \in [s, s+C]} a_m(F). \tag{5.3}$$

6 Boundary Shifts and the Audited Cubature Hierarchy

6.1 Resolution of Intermediate Gaps via Resonant Shell Skipping

In $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$, $\Delta^2 = q^2 - 48q^3 + 1080q^4 - \ldots$, so $a_3(\Delta^2) = \mathbf{-48} < 0$. Relative to its norm, Shell 3 carries an excessively negative moment, driving adjacent weights negative under rigid S_2 anchoring. Skipping Shell 3 or shifting the boundary relieves this constraint.

Theorem 6.1 (Exact Rational Certificates for Strengths $21, 25, 29, 33$)**.** *Our exact rational audit computes strictly positive solutions over the following supports:*

1. ***Strength 21*** **(**$k = 20$**,** $C = 4$**)*:*** $X_{21} = S_2 \cup \{S_4, S_5, S_6, S_7\}$.
2. ***Strength 25*** **(**$k = 24$**,** $C = 7$**)*:*** $X_{25} = S_2 \cup \{S_4, S_5, S_6, S_7, S_8, S_9, S_{10}\}$.
3. ***Strength 29*** **(**$k = 28$**,** $C = 10$**)*:*** $X_{29} = S_2 \cup \{S_5, S_6, \ldots, S_{14}\}$.
4. ***Strength 33*** **(**$k = 32$**,** $C = 14$**)*:*** $X_{33} = S_4 \cup S_5 \cup \cdots \cup S_{18}$.

For each configuration, we certify: $W > 0$*,* $AW = 0$ *in* $\mathbb{Q}^C$*, and* $\Phi_W \neq 0$ *on* $S^0_{k_{\mathrm{next}}+12}(\mathrm{SL}_2(\mathbb{Z}))$*.*

6.2 Explicit Rational Solutions for Early Strengths

To demonstrate exact verification, we report the closed-form rational weight vectors for low degrees:

1. **Strength 15 (**$k = 12$**,** $S_2 \ldots S_3$**):**

$$(W_2, W_3) = \left(1, \ \frac{243}{1024}\right).$$

2. **Strength 17 (**$k = 16$**,** $S_2 \ldots S_4$**):**

$$(W_2, W_3, W_4) = \left(1, \ \frac{5103}{1024}, \ \frac{32}{27}\right).$$

3. **Strength 19 ($k = 18$, $S_2 \dots S_5$):**

$$(W_2, W_3, W_4, W_5) = \left(1,\ \frac{177147}{180224},\ \frac{32}{55},\ \frac{78125}{720896}\right).$$

4. **Strength 21 ($k = 20$, $S_2 \cup \{S_4 \dots S_7\}$):**

$$(W_2, W_4, W_5, W_6, W_7) = \left(1,\ \frac{18425344}{66981555},\ \frac{36865234375}{85355339776},\ \frac{44700093}{208387060},\ \frac{2184859579}{54871289856}\right).$$

5. **Strength 23 ($k = 22$, $S_2 \dots S_7$):**

$$(W_2, \dots, W_7) = \left(1,\ \frac{13657502259}{21237334016},\ \frac{431104}{364563},\ \frac{20556640625}{21237334016},\ \frac{59049}{162028},\ \frac{21750594173}{382272012288}\right).$$

6.3 Midpoint Wave-Packet Centering

Defining the normalized spherical probability mass $w_m = N_m W_m / \sum N_j W_j$, our computations reveal that w_m forms a unimodal traveling wave packet. The peak mass shell m^* closely tracks the spectral envelope scale $k^2/96$.

The spectral scale $k^2/96$ arises naturally as the geometric midpoint of the active window: by Proposition 3.1, the window width scales as $M(k) \approx C(k) \approx k^2/48$. For an active window starting near the origin, the probability mass concentrates symmetrically around the window midpoint $M(k)/2 \approx k^2/96$.

7 The $k \equiv 2 \pmod{12}$ Boundary Branch and Serre Obstruction

7.1 The Serre $M_2 = 0$ Obstruction

When $k = 12m + 2$, the quotient space has modular weight $w = k - 12 = 12(m-1) + 2$. By Serre's theorem, there are no non-trivial modular forms of weight 2:

$$M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}. \tag{7.1}$$

Consequently, in the graded ring $M_*(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{Q}[E_4, E_6]$, one cannot factor out $\Delta^{m-1}\cdot$(weight 2). The maximal admissible discriminant factorization is:

$$\Delta^{m-2} \cdot (\text{weight } 14) = \Delta^{m-2} \cdot E_4^2 E_6. \tag{7.2}$$

This explains why the boundary shift tracks the rate $\Delta m / \Delta k = 1/12$: each increment $\Delta m = 1$ clears one additional q-order of vanishing at the cusp.

7.2 Certified Boundary Minimality

Theorem 7.1 (Certified Boundary Minimality for $k \equiv 2 \pmod{12}$)**.** *For every degree $k \in \{62, 74, 86, 98, 110\}$, our exhaustive predecessor-window audit establishes:*

$$m_{\text{start}}^{\min}(k) = \frac{k-2}{12}. \tag{7.3}$$

Every predecessor starting shell $2 \le s < m_{\text{start}}^{\min}(k)$ fails positivity, while $s = m_{\text{start}}^{\min}(k)$ yields an exact positive rational solution.

8 Deterministic Cusp Form Sign-Change Theorem

Theorem 8.1 (Deterministic Sign Change for Cusp Forms through Weight 122)**.** *Let* $k \in \{62, 74, 86, 98, 110\}$*. For every non-zero cusp form:*

$$F(\tau) \in \Delta^2 \cdot M_{k-12}(\mathrm{SL}_2(\mathbb{Z})) \setminus \{0\}, \tag{8.1}$$

the Fourier coefficients $\{a_m(F)\}$ *cannot have constant sign on the discrete interval:*

$$m \in \left[\frac{k-2}{12}, \frac{k-2}{12} + C(k)\right]. \tag{8.2}$$

That is, every non-zero cusp form takes both strictly positive and strictly negative values within this window.

Proof. For each listed degree k, Table 7.1 provides an exact rational weight vector $W \in \mathbb{Q}^{C+1}_{>0}$ satisfying $AW = 0$ with $\mathrm{rank}(A) = C$. By Theorem 5.1, no non-zero cusp form in $S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z}))$ can have $a_m(F) \geq 0$ for all $m \in [s, s + C]$. Applying the same result to $-F$ proves that F must take both strictly positive and strictly negative values. □

9 Orbit Splitting and Internal Shell Cancellation

9.1 Evaluation of the Canonical Invariant Harmonic Ψ_{12}

Projecting the degree-12 Gegenbauer polynomial $C^{(11)}_{12}(t)$ onto S_2 defines the unique Co_0-invariant harmonic polynomial:

$$\Psi_{12}(x) = \|x\|^{12} \sum_{y \in S_2} C^{(11)}_{12}\left(\frac{x \cdot y}{\|x\|\|y\|}\right). \tag{9.1}$$

The degree-12 Gegenbauer polynomial with $\alpha = 11$ is:

$$\begin{aligned} C^{(11)}_{12}(t) = {} & 2{,}648{,}662{,}016\, t^{12} - 3{,}972{,}993{,}024\, t^{10} + 2{,}128{,}389{,}120\, t^{8} \\ & - 496{,}624{,}128\, t^{6} + 49{,}008{,}960\, t^{4} - 1{,}633{,}632\, t^{2} + 8{,}008. \end{aligned} \tag{9.2}$$

Because the argument $t = \frac{x \cdot y}{\|x\|\|y\|}$ is a scale-invariant cosine, evaluating on the sphere S^{23} yields integer sums. For minimal vectors $u \in S_2$, integer contraction over S_2 yields:

$$\Psi_{12}(S_2) = 1{,}771{,}619{,}850, \qquad \frac{\Psi_{12}(S_3)}{\Psi_{12}(S_2)} = -\frac{9}{16}. \tag{9.3}$$

9.2 Shell 8 Orbit Splitting and Harmonic Sign Inversion

In the intrinsic lattice metric, Shell 8 consists of vectors with squared norm $\|x\|^2 = 2m = 16$. It decomposes under Co_0 into the doubled minimal orbit $\mathcal{O}_{8_A}$ and multiple primitive orbits.

Theorem 9.1 (Conway Orbit Decomposition and Harmonic Sign Inversion)**.** *The shell* S_8 *contains:*

1. *The doubled minimal orbit* $\mathcal{O}_{8_A} = 2 \cdot S_2$ *of cardinality* $|\mathcal{O}_{8_A}| = |S_2| = 196{,}560$*. By linearity of group actions* ($g(2u) = 2g(u)$)*,* $\mathrm{Stab}_{\mathrm{Co}_0}(2u) = \mathrm{Stab}_{\mathrm{Co}_0}(u) \cong 2 \cdot \mathrm{Co}_2$ *[4, Theorem 11, Chapter 10].*

2. *The complement* $S_8 \setminus \mathcal{O}_{8_A}$ *consisting of* $N_8 - 196{,}560 = 814{,}879{,}578{,}240$ *primitive vectors.*

The values of Ψ_{12} on $\mathcal{O}_{8_A}$ and on the representative primitive vector $v_B = \frac{1}{\sqrt{8}}(10, 2^7, 0^{16}) \in S_8$ satisfy:

$$\frac{\Psi_{12}(\mathcal{O}_{8_A})}{\Psi_{12}(S_2)} = +4{,}096 = 2^{12}, \qquad \frac{\Psi_{12}(v_B)}{\Psi_{12}(S_2)} = -\frac{40}{23}. \tag{9.4}$$

Consequently, $\Psi_{12}(\mathcal{O}_{8_A})/\Psi_{12}(v_B) = -11{,}776/5 = -2{,}355.2$.

Proof. Homogeneity gives $\Psi_{12}(2u) = 2^{12}\Psi_{12}(u) = 4{,}096\Psi_{12}(S_2)$.

In the standard octad frame scaled by $\sqrt{8}$, the primitive vector $v_B = \frac{1}{\sqrt{8}}(10, 2^7, 0^{16})$ has squared norm $\|v_B\|^2 = \frac{100+28}{8} = 16$. Its inner products against the 196,560 vectors of S_2 take values $v_B{\cdot}y \in \{\pm 6, \pm 5, \pm 4, \pm 3, \pm 2, \pm 1, 0\}$ with multiplicities $\{8, 256, 2{,}268, 9{,}472, 23{,}608, 39{,}424, 46{,}488\}$.

Evaluating $\sum_{y\in S_2} C_{12}^{(11)}\left(\frac{v_B\cdot y}{8}\right)$ against these multiplicities yields $-\frac{192{,}567{,}375}{256}$. Multiplying by the norm dilation factor $(\|v_B\|/\|u\|)^{12} = (16/4)^6 = 4{,}096$ yields:

$$\Psi_{12}(v_B) = 4{,}096 \times \left(-\frac{192{,}567{,}375}{256}\right) = -3{,}081{,}078{,}000. \tag{9.5}$$

Dividing by $\Psi_{12}(S_2) = 1{,}771{,}619{,}850$ yields $-\frac{3{,}081{,}078{,}000}{1{,}771{,}619{,}850} = -\frac{40}{23}$. □

Remark 9.2 (Multi-Orbit Structure of Shell 8)**.** Because the full Fourier coefficient $a_8(\Delta^2) = +5{,}145{,}888 > 0$ is strictly positive, the total shell sum satisfies $\sum_{x\in S_8}\Psi_{12}(x) = 1{,}011{,}475{,}745{,}280 \times \Psi_{12}(S_2) > 0$. Since $\Psi_{12}(v_B) < 0$, the primitive vectors cannot form a single transitive orbit under Co_0, proving that Shell 8 decomposes into multiple orbits of opposing harmonic signs.

Corollary 9.3 (Internal Shell 8 Annihilation)**.** *Let $\mathcal{O}_{8_B} = \mathrm{Co}_0 \cdot v_B$. Assigning positive scalar weights w_A, w_B cancels degree 12 internally across $\mathcal{O}_{8_A} \cup \mathcal{O}_{8_B}$:*

$$w_A|\mathcal{O}_{8_A}|\Psi_{12}(\mathcal{O}_{8_A}) + w_B|\mathcal{O}_{8_B}|\Psi_{12}(\mathcal{O}_{8_B}) = 0. \tag{9.6}$$

Taking $|\mathcal{O}_{8_B}| = 814{,}879{,}578{,}240$ yields the exact rational ratio:

$$\frac{w_B}{w_A} = \frac{|\mathcal{O}_{8_A}| \cdot 4{,}096}{|\mathcal{O}_{8_B}| \cdot (40/23)} = \frac{196{,}560 \cdot 4{,}096 \cdot 23}{814{,}879{,}578{,}240 \cdot 40} = \frac{64}{112{,}655} \approx \frac{1}{1{,}760.23}. \tag{9.7}$$

10 The Explicit Strength-15 Construction

For a shell or orbit $\mathcal{O}$ of squared norm $2m$, its normalized degree-12 moment contribution is:

$$M(\mathcal{O}) = |\mathcal{O}|(2m)^{-6}\frac{\Psi_{12}(\mathcal{O})}{\Psi_{12}(S_2)}. \tag{10.1}$$

Evaluating on the minimal shells:

$$M_2 = |S_2| \cdot (4)^{-6} = \frac{196{,}560}{4{,}096} = \frac{12{,}285}{256}, \tag{10.2}$$

$$M_3 = |S_3| \cdot (6)^{-6}\left(-\frac{9}{16}\right) = \frac{16{,}773{,}120}{46{,}656}\left(-\frac{9}{16}\right) = \frac{29{,}120}{81}\left(-\frac{9}{16}\right) = -\frac{1{,}820}{9}. \tag{10.3}$$

Theorem 10.1 (Closed-Form Spherical 15-Design on S^{23})**.** *The two-shell configuration $S_2 \cup S_3$ with per-vector shell weights:*

$$W(S_2) = 1, \qquad W(S_3) = \frac{243}{1024} = \frac{3^5}{2^{10}} \tag{10.4}$$

forms an exact spherical 15-design on $N_{\text{total}} = 196{,}560 + 16{,}773{,}120 = 16{,}969{,}680$ distinct spherical points.

Proof. Solving $W(S_2)M_2 + W(S_3)M_3 = 0$ with $W(S_2) = 1$ yields:

$$W(S_3) = \frac{M_2}{-M_3} = \frac{12{,}285/256}{1{,}820/9} = \frac{12{,}285 \times 9}{256 \times 1{,}820} = \frac{110{,}565}{465{,}920} = \frac{243}{1024}. \tag{10.5}$$

Equivalently, in terms of modular Fourier coefficients:

$$W(S_2)(4)^{-6}a_2(\Delta^2)+W(S_3)(6)^{-6}a_3(\Delta^2) = 0 \iff W_2\frac{1}{4096}-W_3\frac{48}{46656} = 0 \iff \frac{W_3}{W_2} = \frac{972}{4096} = \frac{243}{1024}. \tag{10.6}$$

All odd moments vanish by central symmetry, degrees $2 \le k \le 10$ vanish by Venkov's 11-design theorem, and degree 14 vanishes because $M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$. ☐

10.1 Three-Layer Euclidean Harmonic Cancellation

In $\mathbb{R}^{24}$, regarding S_2, S_3, and $\mathcal{O}_{8_A}$ as three distinct radial layers ($\|x\|^2 \in \{4, 6, 16\}$), radial scaling gives $M_{8_A} = |\mathcal{O}_{8_A}|(16)^{-6}(4096) = 196{,}560 \cdot 2^{-24} \cdot 2^{12} = \frac{12{,}285}{256} = M_2$. Setting unit weights $W_2 = W_{8_A} = 1$:

$$2M_2 + W_3M_3 = 0 \implies W_3 = \frac{2M_2}{-M_3} = \frac{243}{512} = \frac{3^5}{2^9}. \tag{10.7}$$

This provides exact degree-12 harmonic cancellation across three concentric radii in $\mathbb{R}^{24}$.

11 Intertwiner Elasticity

Let $u \in S_2$ be a minimal vector. In scaled coordinates where $\|u\|^2 = 32$, the stabilizer subgroup in Co_0 is:

$$K = \mathrm{Stab}_{\mathrm{Co}_0}(u) \cong 2 \cdot \mathrm{Co}_2\,. \tag{11.1}$$

Under K, the 196,560 vectors of S_2 decompose into exactly 7 orbits characterized by $u \cdot v \in \{\pm 32, \pm 16, \pm 8, 0\}$:

$u \cdot v$:	± 32	± 16	± 8	0
Orbit Size :	1 each	4,600 each	47,104 each	93,150

(11.2)

The space of K-equivariant tangent vector fields on S_2 is:

$$\mathcal{M} = \mathrm{Hom}_K(V_{\mathrm{nat}}, T_{S_2}S^{23}). \tag{11.3}$$

On each of the 5 non-polar orbits ($s \in \{\pm 16, \pm 8, 0\}$), the space of equivariant fields has dimension 2, spanned by:

- **Transverse field:** $W_1(z, v) = z - \frac{z \cdot v}{32}v$,
- **Longitudinal field:** $W_2(z, v) = (z \cdot v)\left[u - \frac{u \cdot v}{32}v\right]$.

On the 2 polar orbits ($s = \pm 32$), $v = \pm u \implies W_2 \equiv 0$, leaving 1 field each. Thus:

$$\dim \mathcal{M} = 5 \times 2 + 2 \times 1 = 12. \tag{11.4}$$

Proposition 11.1 (Rotational Goldstone Mode and Parity Splitting)**.** *For any smooth radial energy functional $E(X) = \sum_{x \neq y} F(\|x - y\|^2)$:*

1. *The rotational field $\Phi_{\mathrm{rot},z}(v) = (z \cdot v)u - (u \cdot v)z$ satisfies $H\Phi_{\mathrm{rot}} = 0$, spanning the kernel $\ker(H) = \mathrm{span}\{c_{\mathrm{rot}}\}$.*
2. *The antipodal pullback $(P\Phi)(z, v) = \Phi(z, -v)$ satisfies $P^2 = I$ and $[P, H] = 0$, decomposing $\mathcal{M}$ into 6 even (+1) and 6 odd (−1) modes, with $P(c_{\mathrm{rot}}) = -c_{\mathrm{rot}}$.*

12 Completely Monotonic Potentials and the Local Hessian

A potential $F : (0, \infty) \to \mathbb{R}$ is completely monotonic if $(-1)^n F^{(n)}(r) \geq 0$ for all $n \geq 0$. By the Bernstein–Widder theorem:

$$F(r) = \int_{[0,\infty)} e^{-tr}\, d\mu(t), \qquad d\mu \geq 0. \tag{12.1}$$

Constant potentials ($t = 0$) have identically zero Hessian. For non-constant potentials, $\mu((0, \infty)) > 0$.

Proposition 12.1 (Conditional Local Positivity Criterion)**.** *On the 12-dimensional intertwiner space $\mathcal{M}$, the Hessian for a Gaussian $G_t(r) = e^{-tr}$ ($t > 0$) decomposes as $A(G_t) = \lambda_S(G_t)I - K_{\tan}(G_t)$, where the diagonal curvature:*

$$\lambda_S(G_t) = \sum_s \left[4t^2 e^{-tU_s} c_s + 2te^{-tU_s} \frac{s}{32} \operatorname{count}(s)\right] \tag{12.2}$$

satisfies the exact hyperbolic-sine identity:

$$\sum_{s \in \{\pm 32, \pm 16, \pm 8, 0\}} 2te^{-tU_s} \frac{s}{32} \operatorname{count}(s) = 4te^{-64t} \left[\sinh(64t) + 2{,}300 \sinh(32t) + 11{,}776 \sinh(16t)\right] > 0 \tag{12.3}$$

for all $t > 0$.

Condition H: *Assume that for all $t > 0$, the operator norm of the tangential repulsion tensor restricted to the orthogonal complement of the Goldstone mode satisfies:*

$$\|K_{\tan}(G_t)\|_{\mathcal{M} \ominus \mathbb{R}c_{\mathrm{rot}}} < \lambda_S(G_t). \tag{12.4}$$

Under Condition H, every non-constant completely monotonic potential F has a strictly positive Hessian on $\mathcal{M} \ominus \mathbb{R}c_{\mathrm{rot}}$:

$$v^T H_F v = \int_{(0,\infty)} \left(v^T A(G_t) v\right) d\mu(t) > 0 \qquad \forall v \in \mathcal{M} \setminus \mathbb{R}c_{\mathrm{rot}}. \tag{12.5}$$

13 Point Counts and Asymptotic Delsarte Comparison

The number of conditions scales as $C(t) \sim t^2/48 \implies M(t) = \Theta(t^2)$. The boundary shift $m_{\mathrm{start}}(t) \sim t/12 = \Theta(t)$ is subdominant: $\lim_{t\to\infty} \frac{m_{\mathrm{start}}(t)}{M(t)} = 0$. Since $N_m = \Theta(m^{11})$, the total vector count satisfies:

$$N_{\mathrm{total}}(t) = \sum_{m=m_{\mathrm{start}}}^{m_{\mathrm{start}}+M-1} N_m \asymp M(t)^{12} \asymp (t^2)^{12} = \Theta(t^{24}). \tag{13.1}$$

In dimension $d = 24$, the general positive cubature lower bound is $\Theta(t^{23})$. The multi-shell construction achieves an exponent of 24, which is within an additional factor of $\mathcal{O}(t)$ of the lower bound, matching the Korevaar–Meyers upper bound scale $\mathcal{O}(t^d) = \mathcal{O}(t^{24})$.

14 Computational Certification Protocol

All computational certificates in this paper are exact rational statements verified via FLINT [6]. The verification package contains:

1. The exact monomial basis $F_b = \Delta^2 E_4^{a_b} E_6^{b_b}$ of $S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z}))$.

2. The exact rational moment matrix $A \in \mathbb{Q}^{C\times(C+1)}$.
3. A certificate proving $\mathrm{rank}(A) = C$.
4. The exact rational solution vector $W \in \mathbb{Q}^{C+1}_{>0}$ with certified positive coordinates.
5. Exact rational verification that the residual vector satisfies $AW = 0$ identically in $\mathbb{Q}^C$.
6. An exact witness $F \in S^0_{k_{\mathrm{next}}+12}(\mathrm{SL}_2(\mathbb{Z}))$ proving $L_W(F) \neq 0$.
7. For boundary minimality, exact negative-weight certificates for every predecessor window $s < m^{\min}_{\mathrm{start}}(k)$.
8. Cryptographic SHA-256 hashes of all input matrices and output weight vectors.

15 Scope of the Results

To maintain strict scientific standards, we delineate our results into three categories:

15.1 Proved Structural Results

- The Reynolds shell annihilation theorem (Proposition 2.1).
- The cusp space isomorphism $S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})) \cong M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$ (Lemma 2.2).
- The exact asymptotic formula $C(k) = k^2/48 - k/4 + \delta(k)$ (Proposition 3.1).
- The Farkas–Gordan modular alternative theorem (Theorem 5.1).
- The closed-form spherical 15-design on $S_2 \cup S_3$ with weights $(1, 243/1024)$ (Theorem 10.1).

15.2 Certified Computational Results

- Exact rational cubatures for all audited degrees in Table 6.1 through strength 111.
- Predecessor-window failures and boundary minimality for $k \in \{62, 74, 86, 98, 110\}$ (Theorem 7.1).
- Unconditional cusp form sign changes up to weight 122 (Theorem 8.1).
- Shell 8 orbit splitting and internal annihilation with ratio 64/112,655 (Theorem 9.1).

15.3 Conjectures and Open Questions

- The extension of $m^{\min}_{\mathrm{start}}(k) = (k-2)/12$ to all $k \equiv 2 \pmod{12}$ as $k \to \infty$.
- The all-weight modular Chebyshev sign-change property.
- Complete classification of Co_0-sub-orbits across all higher Leech shells.

16 Conclusion

We have established an exact rational multi-shell cubature framework on the Leech lattice Λ_{24}. By leveraging Reynolds group averaging and the isomorphism $S^0_{k+12}(\mathrm{SL}_2(\mathbb{Z})) \cong M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$, the infinite system of degree-k harmonic moment conditions collapses to a finite rational linear system.

Our exact FLINT computations certify positive rational spherical cubatures through strength 111, proving that intermediate gaps are resolved by resonant shell skipping. Through modular cone duality, our certified positive weights provide an unconditional proof of finite-window sign changes for cusp forms up to weight 122. Finally, Co_0-orbit splitting reveals an exact harmonic sign inversion on Shell 8, enabling internal spherical designs and compressed cubature hierarchies. Unified master exact audit file is available at:

https://srfp311t1.com/leech_cubature_111.py

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Table 6.1: Certified exact rational audit of multi-shell cubatures through degree $k = 110$.

k	t	C	M	Active Window	Positive	$AW = 0$	First omitted	Max Denom	Peak Mass
12	15	1	2	$S_2 \dots S_3$	YES	True	True	4d (11b)	S_3
14	15	1	2	$S_2 \dots S_3$	YES	True	True	4d (11b)	S_3
16	17	2	3	$S_2 \dots S_4$	YES	True	True	4d (11b)	S_4
18	19	3	4	$S_2 \dots S_5$	YES	True	True	6d (20b)	S_5
20	21	4	5	$S_2 \cup \{S_4 \dots S_7\}$	YES	True	True	9d (28b)	S_6
22	23	5	6	$S_2 \dots S_7$	YES	True	True	12d (39b)	S_6
24	25	7	8	$S_2 \cup \{S_4 \dots S_{10}\}$	YES	True	True	16d (52b)	S_8
26	27	8	9	$S_2 \dots S_{10}$	YES	True	True	15d (50b)	S_8
28	29	10	11	$S_2 \cup \{S_5 \dots S_{14}\}$	YES	True	True	20d (65b)	S_{10}
30	31	12	13	$S_2 \dots S_{14}$	YES	True	True	22d (72b)	S_{11}
32	33	14	15	$S_4 \dots S_{18}$	YES	True	True	26d (84b)	S_{12}
34	35	16	17	$S_2 \dots S_{18}$	YES	True	True	30d (99b)	S_{13}
38	39	21	22	$S_2 \dots S_{23}$	YES	True	True	40d (131b)	S_{16}
40	41	24	25	$S_2 \dots S_{26}$	YES	True	True	44d (144b)	S_{18}
42	43	27	28	$S_3 \dots S_{30}$	YES	True	True	50d (164b)	S_{19}
46	47	33	34	$S_2 \dots S_{35}$	YES	True	True	64d (212b)	S_{22}
50	51	40	41	$S_2 \dots S_{42}$	YES	True	True	81d (267b)	S_{26}
54	55	48	49	$S_5 \dots S_{53}$	YES	True	True	88d (292b)	S_{30}
58	59	56	57	$S_2 \dots S_{58}$	YES	True	True	121d (402b)	S_{33}
62	63	65	66	$S_5 \dots S_{70}$	YES	True	True	154d (509b)	S_{37}
66	67	75	76	$S_6 \dots S_{81}$	YES	True	True	178d (590b)	S_{40}
70	71	85	86	$S_4 \dots S_{89}$	YES	True	True	218d (723b)	S_{47}
74	75	96	97	$S_6 \dots S_{102}$	YES	True	True	241d (799b)	S_{52}
78	79	108	109	$S_7 \dots S_{115}$	YES	True	True	299d (993b)	S_{57}
82	83	120	121	$S_7 \dots S_{127}$	YES	True	True	325d (1077b)	S_{62}
86	87	133	134	$S_7 \dots S_{140}$	YES	True	True	401d (1332b)	S_{68}
90	91	147	148	$S_2 \dots S_{149}$	YES	True	True	437d (1452b)	S_{74}
94	95	161	162	$S_5 \dots S_{166}$	YES	True	True	808d (2683b)	S_{80}
98	99	176	177	$S_8 \dots S_{184}$	YES	True	True	1161d (3857b)	S_{86}
102	103	192	193	$S_{10} \dots S_{202}$	YES	True	True	1424d (4730b)	S_{91}
106	107	208	209	$S_8 \dots S_{216}$	YES	True	True	1403d (4661b)	S_{98}
110	111	225	226	$S_9 \dots S_{234}$	YES	True	True	1580d (5248b)	S_{104}

Table 6.2: Observed mass-peak locations versus the spectral midpoint scale $k^2/96$.

Strength t	Shells M	Active Window	Observed Peak m^*	Midpoint Scale $k^2/96$
23	6	$S_2 \dots S_7$	S_6	5.0
35	17	$S_2 \dots S_{18}$	S_{13}	12.0
51	41	$S_2 \dots S_{42}$	S_{26}	26.0
71	86	$S_4 \dots S_{89}$	S_{47}	51.0
91	148	$S_2 \dots S_{149}$	S_{74}	84.4
103	193	$S_{10} \dots S_{202}$	S_{91}	108.4
107	209	$S_8 \dots S_{216}$	S_{98}	117.0
111	226	$S_9 \dots S_{234}$	S_{104}	126.0

Table 7.1: Exhaustive predecessor-window audit for the $k \equiv 2 \pmod{12}$ branch.

k	C	M	m_{start}	Predecessor Audits ($s < m_{\text{start}}$)	Certified Window
62	65	66	**5**	$s = 2$ (NO(1)), $s = 3$ (NO(65)), $s = 4$ (NO(13))	$S_5 \dots S_{70}$
74	96	97	**6**	$s = 2 \dots 4$ (NO(1–2)), $s = 5$ (NO(96))	$S_6 \dots S_{102}$
86	133	134	**7**	$s = 2 \dots 5$ (NO(4–133)), $s = 6$ (NO(2))	$S_7 \dots S_{140}$
98	176	177	**8**	$s = 2 \dots 7$ (NO(3–175))	$S_8 \dots S_{184}$
110	225	226	**9**	$s = 2 \dots 8$ (NO(2–223))	$S_9 \dots S_{234}$

Table 13.1: Comparison of certified Leech cubatures against the equal-weight DGS bound.

Strength t	DGS Bound N_{DGS}	Active Leech Configuration	Points Used N_{total}	Ratio $\frac{N_{\mathrm{total}}}{N_{\mathrm{DGS}}}$
11	**196,560**	S_2	**196,560**	**1.0000** (TIGHT)
15	**4,071,600**	$S_2 \cup S_3$	**16,969,680**	**4.17** (NEAR-OPT.)
17	15,777,450	$S_2 \dots S_4$	415,003,680	26.3
19	56,097,600	$S_2 \dots S_5$	5,044,384,800	89.9
21	185,122,080	$S_2 \cup \{S_4 \dots S_7\}$	226,935,203,040	1,226
23	572,195,520	$S_2 \dots S_7$	226,951,976,160	397
91	1.60×10^{18}	$S_2 \dots S_{149}$	9.85×10^{26}	6.1×10^{8}
111	$\mathbf{6.90 \times 10^{19}}$	$S_9 \dots S_{234}$	$\mathbf{2.19 \times 10^{29}}$	$\mathbf{3.2 \times 10^{9}}$