

THE ASYMPTOTIC HORIZON OF LEECH LATTICE CUBATURES: ARCHIMEDEAN DAMPING, SPECTRAL SCALING, AND FUNCTIONAL-ANALYTIC LIMITING CRITERIA

SRFP311T1 COLLABORATION

ABSTRACT. We develop a conditional functional-analytic framework for extracting subsequential limits of suitably damped multi-shell cubature measures supported on the 24-dimensional Leech lattice Λ_{24} . Building upon the finite-dimensional sparse cubatures established in Parts I and II, we analyze the mathematical requirements governing the infinite-dimensional limit ($k \rightarrow \infty$).

First, we analyze the ℓ^1 -norm divergence obstacle (Gate 1). Introducing the *Archimedean Spectral Damping Kernel*

$$\mathcal{K}_{s,\beta}(m) = (2m)^{-s} \exp(-2\pi\sqrt{m}/\beta),$$

we prove that, under an explicit polynomial-exponential growth hypothesis on the rescaled cubature variables

$$\widetilde{W}_m^{(k)} = W_m^{(k)} (2m)^{-k/2},$$

the damped discrete measures

$$\mu_k = \sum_{m \in S_k} \widetilde{W}_m^{(k)} \mathcal{K}_{s,\beta}(m) \delta_m$$

remain uniformly bounded in total variation on $\ell^1(\mathbb{N}_{\geq 2})$ for all $s > \alpha + 1$.

Second, we analyze the experimental *Desert Phenomenon* observed in Part II ($\Delta m = 117$ at $k = 114$). We formulate the *Spectral Horizon Conjecture*

$$m^*(k) \asymp k^2/96,$$

motivated by a saddle-point transition in high-weight modular contour integrals. The saddle-point calculation establishes the relevant k/m radial scaling, while the specific constant $1/96$ remains conjectural. We also clarify the analytic distinction between horizontal Hecke Sato–Tate distributions and vertical high-weight trace asymptotics.

Third, by embedding the discrete shell measures into the Banach dual pair $(\ell^1(\mathbb{N}_{\geq 2}), c_0(\mathbb{N}_{\geq 2}))$, we prove the *Escape of Outer Mass Theorem*: weak-* accumulation measures in the topology $\sigma(\ell^1, c_0)$ are determined strictly by finite-index shells, while mass escaping to infinity evaluates to zero on the predual.

Finally, we identify the structural geometric reasons why finite-dimensional modular cone duality on Λ_{24} , by itself, does not furnish an implication concerning the positive definiteness of the Guinand–Weil functional.

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1. INTRODUCTION AND THE PROBLEM OF THE CONTINUUM LIMIT

The study of discrete approximations of continuous spherical integrals via lattice configurations constitutes a central theme in geometric analysis, sphere-packing theory, and automorphic forms. In Parts I and II of this series [9, 10], we developed constructions of spherical t -designs on the unit sphere $S^{23} \subset \mathbb{R}^{24}$ using concentric shells

$$S_m = \{x \in \Lambda_{24} : \|x\|^2 = 2m\}.$$

After averaging under the Conway group $\text{Co}_0 = \text{Aut}(\Lambda_{24})$, the corresponding cubature conditions reduce to a finite-dimensional linear feasibility problem.

Specifically, for each even degree $k \geq 12$, the discrete cubature condition is

$$\sum_{m \in S_k} W_m^{(k)} v_{\leq k}(m) = \mathbf{0} \in \mathbb{R}^{D_k}, \quad W_m^{(k)} > 0, \quad (1.1)$$

where

$$D_k = \sum_{j=6}^{k/2} \dim M_{2j-12}(\text{SL}_2(\mathbb{Z})) \approx \frac{k^2}{48},$$

and $v_{\leq k}(m)$ is the cumulative evaluation vector across the direct sum of cusp spaces

$$S_{2j+12}^{(2)}(\text{SL}_2(\mathbb{Z})) := \{f \in S_{2j+12}(\text{SL}_2(\mathbb{Z})) : \text{ord}_{q=0}(f) \geq 2\} \cong \Delta^2 M_{2j-12}(\text{SL}_2(\mathbb{Z})). \quad (1.2)$$

1.1. The Stability Gates for Infinite-Dimensional Limits. In Part II [10], we identified five independent functional-analytic stability criteria required before finite-dimensional polyhedral certificates can be interpreted as continuous or infinite-dimensional positivity statements:

Gate 1: Uniform Total Variation Boundedness: Establish

$$\sup_k \sum_{m \in S_k} W_m^{(k)} \mathcal{K}(m) < \infty$$

under an explicit damping kernel $\mathcal{K}(m)$.

Gate 2: Well-Defined Ambient Embedding: Specify a fixed topological vector space E and dual E^* into which all discrete measures μ_k are embedded.

Gate 3: Weak-* Compactness and Accumulation: Extract a weak-* limit $\mu_\infty \in E^*$ via the Banach–Alaoglu theorem in $\sigma(E^*, E)$.

Gate 4: Test-Function Compatibility: Verify that the target cusp-form evaluation functionals belong to the predual test space E .

Gate 5: Non-Triviality / Mass Preservation: Control the asymptotic mass allocation sufficiently to prevent collapse to the trivial zero functional:

$$\liminf_{k \rightarrow \infty} \langle \mu_k, \mathbf{1} \rangle > 0.$$

1.2. The Growth Obstacle in Raw Cubatures. In Part II, numerical evaluation of unweighted cubature solutions across degrees $12 \leq k \leq 32$ revealed rapid growth in the unnormalized ℓ^1 norm:

$$\|W^{(12)}\|_1 \approx 1.24, \quad \|W^{(20)}\|_1 \approx 5.29, \quad \|W^{(28)}\|_1 \approx 24.88, \quad \|W^{(32)}\|_1 \approx 199.73. \quad (1.3)$$

The growth ratio accelerates to

$$\frac{\|W^{(32)}\|_1}{\|W^{(28)}\|_1} \approx 8.03.$$

The mathematical origin of this growth is structural: the radial projection factors $(2m)^{-k/2}$ decay rapidly as m increases, requiring outer shells to carry large weights W_m in order to balance the low-order anchor shell S_{12} .

In this paper, we analyze the conditions under which an Archimedean damping kernel regularizes this growth, formulate the saddle-point scaling behind the observed desert horizon, and establish the functional-analytic framework for subsequential limits of the resulting measures on $\ell^1(\mathbb{N}_{\geq 2})$.

2. ARCHIMEDEAN DAMPING AND TOTAL VARIATION BOUNDS

Let

$$\mathbb{N}_{\geq 2} = \{2, 3, 4, \dots\}$$

denote the discrete index space of Leech shells. We equip $\ell^1(\mathbb{N}_{\geq 2})$ with its standard norm

$$\|\mu\|_{\text{TV}} = \sum_{m=2}^{\infty} |\mu(m)|.$$

Definition 2.1 (Archimedean Spectral Damping Kernel). For parameters $s \in \mathbb{R}$ with $s > 12$ and $\beta > 0$, define

$$\mathcal{K}_{s,\beta}(m) := (2m)^{-s} \exp\left(-\frac{2\pi\sqrt{m}}{\beta}\right), \quad m \geq 2. \quad (2.1)$$

In the linear feasibility problem

$$A_{\leq k} W^{(k)} = \mathbf{0},$$

the matrix entries are

$$A_{b,m} = (2m)^{-k/2} a_m(F_b).$$

This motivates introducing the rescaled cubature variables

$$\widetilde{W}_m^{(k)} := W_m^{(k)} (2m)^{-k/2}. \quad (2.2)$$

For each target degree k , let $(W^{(k)}, \mathcal{S}_k)$ be a non-negative cubature solution. We associate to it the regularized discrete measure

$$\mu_k^{(s,\beta)} := \sum_{m \in \mathcal{S}_k} \widetilde{W}_m^{(k)} \mathcal{K}_{s,\beta}(m) \delta_m \in \ell^1(\mathbb{N}_{\geq 2}), \quad (2.3)$$

where δ_m denotes the standard Dirac point mass at $m \in \mathbb{N}_{\geq 2}$.

To establish uniform boundedness in $\ell^1(\mathbb{N}_{\geq 2})$, we formalize the necessary growth hypothesis directly on the rescaled variables.

Hypothesis 2.2 (Polynomial-Exponential Rescaled Weight Bound). There exist constants $C_0 > 0$, $\alpha \geq 0$, and

$$\gamma < \frac{2\pi}{\beta}$$

such that for all degrees k and all active shells $m \in \mathcal{S}_k$,

$$\widetilde{W}_m^{(k)} \leq C_0 m^\alpha e^{\gamma\sqrt{m}}.$$

Theorem 2.3 (Conditional Resolution of Gate 1). *Assume Hypothesis 2.2. Then for any $s > \alpha + 1$, the regularized measures $\{\mu_k^{(s,\beta)}\}$ are uniformly bounded in total variation:*

$$\sup_{k \geq 12} \|\mu_k^{(s,\beta)}\|_{\text{TV}} \leq M(s, \beta) < \infty. \quad (2.4)$$

Proof. Substituting Hypothesis 2.2 into the total variation sum gives

$$\begin{aligned} \|\mu_k^{(s,\beta)}\|_{\text{TV}} &= \sum_{m \in \mathcal{S}_k} \widetilde{W}_m^{(k)} (2m)^{-s} \exp\left(-\frac{2\pi\sqrt{m}}{\beta}\right) \\ &\leq C_0 2^{-s} \sum_{m \in \mathcal{S}_k} m^{\alpha-s} \exp\left(-\left(\frac{2\pi}{\beta} - \gamma\right)\sqrt{m}\right). \end{aligned} \quad (2.5)$$

Set

$$\epsilon := \frac{2\pi}{\beta} - \gamma > 0.$$

Since the summand is non-negative,

$$\sum_{m \in \mathcal{S}_k} m^{\alpha-s} e^{-\epsilon\sqrt{m}} \leq \sum_{m=2}^{\infty} m^{\alpha-s} e^{-\epsilon\sqrt{m}}.$$

The latter series converges because of its exponential decay in $\sqrt{m}$. Indeed,

$$\int_1^{\infty} x^{\alpha-s} e^{-\epsilon\sqrt{x}} dx = 2 \int_1^{\infty} u^{2(\alpha-s)+1} e^{-\epsilon u} du < \infty. \quad (2.6)$$

Thus the right-hand side is bounded by a finite constant independent of k , which proves (2.4). $\square$

Remark 2.4. Hypothesis 2.2 is a substantive mathematical assumption. Non-negative least-squares solvers, including S-NNMC, guarantee $\widetilde{W}_m^{(k)} \geq 0$ but do not automatically provide a uniform upper bound of the form stated above. Deriving such a bound directly from the Karush–Kuhn–Tucker conditions or from the dual cone geometry remains an open problem.

3. THE SPECTRAL HORIZON CONJECTURE

In Part II [10], numerical experiments revealed an exclusion gap, referred to there as the *Desert Phenomenon*:

- At $k = 114$ ($C = 243$), the active support was $S_{12} \cup \{S_{129}, \dots, S_{265}\}$, with gap $\Delta m = 117$.
- At $k = 120$ ($C = 271$), the active support was $S_{12} \cup \{S_{89}, \dots, S_{295}\}$, with gap $\Delta m = 77$.

3.1. Saddle-Point Asymptotic Scaling. The proposed scale

$$m^*(k) \asymp \frac{k^2}{96}$$

is motivated by the asymptotic behavior of the modular basis

$$F_{a,b} = \Delta^2 E_4^a E_6^b, \quad 4a + 6b = k - 12.$$

The corresponding modular weight is $k + 12$.

By Cauchy's integral formula, the m -th Fourier coefficient is

$$a_m(F_{a,b}) = \frac{1}{2\pi i} \oint_{|q|=r} \Delta(q)^2 E_4(q)^a E_6(q)^b q^{-m-1} dq. \quad (3.1)$$

Writing $q = e^{-2\pi u}$ for $u > 0$, the corresponding logarithmic phase along the positive real axis has the form

$$\Phi_k(u) = a \log E_4(e^{-2\pi u}) + b \log E_6(e^{-2\pi u}) + 2 \log \Delta(e^{-2\pi u}) + 2\pi m u.$$

Under modular inversion $\tau \mapsto -1/\tau$, the large modular weight contributes a logarithmic term of order

$$\frac{k}{2} \log(1/u).$$

Thus, schematically, for small u one obtains

$$\Phi_k(u) \approx \frac{k}{2} \log\left(\frac{1}{u}\right) + \frac{c_1}{u} + 2\pi m u, \quad (3.2)$$

where the constant c_1 depends on the precise modular factors under consideration.

Differentiating the displayed model phase gives

$$\Phi'_k(u) = -\frac{k}{2u} - \frac{c_1}{u^2} + 2\pi m. \quad (3.3)$$

When the first and third terms dominate, the associated radial scale is

$$u_0 \asymp \frac{k}{m}. \quad (3.4)$$

This calculation therefore identifies a natural k/m saddle-point scaling. The more specific numerical transition scale $m^*(k) \asymp k^2/96$ is motivated by the observed cone geometry and numerical data; deriving the constant $1/96$ rigorously from the full modular saddle-point analysis remains open.

Conjecture 3.1 (Spectral Horizon Scale). *Let $k \geq 12$. We conjecture that the geometry of the modular evaluation vectors $v_{\leq k}(m)$ undergoes a phase transition at a critical scale*

$$m^*(k) \asymp \frac{k^2}{96}.$$

More precisely, we conjecture:

- (a) **Low-Frequency Phase Locking** ($m \ll k^2/96$): *the relevant saddle lies in a cusp-dominated regime, in which the Fourier expansions are expected to exhibit comparatively restricted angular variation and the evaluation vectors remain confined to an acute region of the associated cone geometry.*
- (b) **High-Frequency Angular Dispersion** ($m \gg k^2/96$): *the relevant saddle moves toward the modular bulk, where the full oscillatory structure of the modular factors is expected to provide the transverse angular dispersion needed for the evaluation vectors to surround the origin.*

Remark 3.2 (Clarification on Sato–Tate Literature). Horizontal Sato–Tate results describe the distribution of normalized Hecke eigenvalues at varying primes for a fixed automorphic form, whereas the limit relevant here is vertical: the modular weight tends to infinity while the Fourier index may remain fixed or vary on a coupled scale. These are different asymptotic regimes. Accordingly, horizontal Sato–Tate results do not directly establish the high-weight asymptotics required for the present saddle-point problem; trace-formula and high-weight coefficient asymptotics are the more directly relevant tools.

4. THE LIMITING MEASURE ON THE SEQUENCE SPACE $\ell^1(\mathbb{N}_{\geq 2})$

To avoid topological type mismatches, the discrete measures $\mu_k^{(s,\beta)}$ are embedded into a single fixed Banach dual pair rather than into the varying finite-dimensional spaces $\mathbb{R}^{D_k}$ or the continuous modular curve $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$.

4.1. The Sequence Dual Pair. We take

$$E = c_0(\mathbb{N}_{\geq 2})$$

and identify its Banach dual canonically with

$$E^* = \ell^1(\mathbb{N}_{\geq 2}).$$

Explicitly,

$$E^* = \ell^1(\mathbb{N}_{\geq 2}) = \left\{ \mu : \mathbb{N}_{\geq 2} \rightarrow \mathbb{R} : \sum_{m=2}^{\infty} |\mu(m)| < \infty \right\}, \quad (4.1)$$

with norm

$$\|\mu\|_{\text{TV}} = \sum_{m=2}^{\infty} |\mu(m)|.$$

The predual is

$$E = c_0(\mathbb{N}_{\geq 2}) = \left\{ \phi : \mathbb{N}_{\geq 2} \rightarrow \mathbb{R} : \lim_{m \rightarrow \infty} \phi(m) = 0 \right\}, \quad (4.2)$$

equipped with

$$\|\phi\|_{\infty} = \sup_{m \geq 2} |\phi(m)|.$$

We equip $\ell^1(\mathbb{N}_{\geq 2})$ with the weak-* topology

$$\sigma(\ell^1, c_0).$$

Theorem 4.1 (Weak-* Compactness and Accumulation). *Let $s > \alpha + 1$ and $\beta > 0$. Assume Hypothesis 2.2. Then the sequence of regularized measures $\{\mu_k^{(s,\beta)}\}_{k \geq 12}$ is contained in a weak-* compact ball in $\ell^1(\mathbb{N}_{\geq 2})$. Consequently, there exists a subsequence $\{k_j\}$ and a non-negative limiting measure $\mu_{\infty} \in \ell^1(\mathbb{N}_{\geq 2})$ such that*

$$\mu_{k_j}^{(s,\beta)} \xrightarrow{*} \mu_{\infty} \quad \text{in } \sigma(\ell^1, c_0). \quad (4.3)$$

That is, for every $\phi \in c_0(\mathbb{N}_{\geq 2})$,

$$\lim_{j \rightarrow \infty} \sum_{m=2}^{\infty} \mu_{k_j}^{(s,\beta)}(m) \phi(m) = \sum_{m=2}^{\infty} \mu_{\infty}(m) \phi(m). \quad (4.4)$$

Proof. By Theorem 2.3,

$$\sup_k \|\mu_k^{(s,\beta)}\|_{\text{TV}} \leq M(s, \beta) < \infty.$$

Hence the sequence lies in the closed ball of radius $M(s, \beta)$ in $E^* = \ell^1$. By the Banach–Alaoglu theorem this ball is compact in the weak-* topology $\sigma(E^*, E)$. Since $c_0(\mathbb{N}_{\geq 2})$ is separable, this weak-* compact ball is metrizable and therefore sequentially compact. Thus a weak-* convergent subsequence exists.

Each $\mu_k^{(s,\beta)}$ is non-negative. The positive cone of ℓ^1 is weak-* closed because

$$\mu(m) = \langle \mu, \delta_m \rangle$$

and each coordinate functional δ_m belongs to c_0 . Hence the limit μ_{∞} is also non-negative. $\square$

4.2. The Escape of Outer Mass. The constant sequence

$$\mathbf{1} = (1, 1, 1, \dots)$$

does not belong to $c_0(\mathbb{N}_{\geq 2})$. Consequently, weak-* convergence in $\sigma(\ell^1, c_0)$ does not preserve total mass:

$$\mu_k(\mathbf{1}) \not\rightarrow \mu_\infty(\mathbf{1})$$

in general.

This phenomenon should be interpreted as *escape of mass relative to the chosen topology*: mass moving to indices $m \rightarrow \infty$ becomes invisible to every test sequence in c_0 .

Theorem 4.2 (Escape of Outer Mass). *Let $(W^{(k)}, \mathcal{S}_k)$ be a sequence of cubature solutions whose active supports satisfy*

$$\mathcal{S}_k \subseteq \{12\} \cup [m_2(k), \infty), \quad m_2(k) \rightarrow \infty.$$

Assume, in addition, that

$$\sup_k \|\mu_k^{(s, \beta)}\|_{\text{TV}} < \infty.$$

Then there exists a subsequence $\{k_j\}$ along which the scalar anchor mass converges,

$$c_{12} := \lim_{j \rightarrow \infty} \widetilde{W}_{12}^{(k_j)} \mathcal{K}_{s, \beta}(12). \quad (4.5)$$

Every weak- accumulation point of this subsequence in $\sigma(\ell^1, c_0)$ is*

$$\mu_\infty = c_{12} \delta_{12}. \quad (4.6)$$

Thus all mass carried by the high-frequency cluster $m \geq m_2(k)$ escapes to infinity relative to the weak- topology $\sigma(\ell^1, c_0)$.*

Proof. Let $\phi \in c_0(\mathbb{N}_{\geq 2})$. For any $\epsilon > 0$, choose N_0 such that

$$|\phi(m)| < \epsilon \quad (m \geq N_0).$$

Since $m_2(k) \rightarrow \infty$, there exists j_0 such that

$$m_2(k_j) \geq N_0 \quad (j \geq j_0).$$

For such j ,

$$\text{supp}(\mu_{k_j}) \setminus \{12\} \subseteq [m_2(k_j), \infty) \subseteq [N_0, \infty).$$

Therefore

$$\begin{aligned} \left| \sum_{m \geq 13} \mu_{k_j}^{(s, \beta)}(m) \phi(m) \right| &= \left| \sum_{m \geq m_2(k_j)} \mu_{k_j}^{(s, \beta)}(m) \phi(m) \right| \\ &\leq \epsilon \sum_{m \geq m_2(k_j)} \mu_{k_j}^{(s, \beta)}(m) \\ &\leq \epsilon M, \end{aligned} \quad (4.7)$$

where

$$M = \sup_k \|\mu_k^{(s, \beta)}\|_{\text{TV}}.$$

Taking $j \rightarrow \infty$ and then $\epsilon \rightarrow 0$ shows that the outer cluster contributes zero to the weak-* pairing. The anchor contribution converges to

$$c_{12} \phi(12)$$

by (4.5). Hence

$$\lim_{j \rightarrow \infty} \langle \mu_{k_j}^{(s, \beta)}, \phi \rangle = c_{12} \phi(12) = \langle c_{12} \delta_{12}, \phi \rangle.$$

Thus every weak-* accumulation point is $c_{12}\delta_{12}$. $\square$

5. DUALITY COMPARISON: THE MODULAR CONE VS. GUINAND–WEIL POSITIVITY

The existence of positive annihilating cones on the Leech lattice raises the question of whether this framework can be transferred to the Guinand–Weil explicit formula to deduce the Riemann Hypothesis (RH).

The relevant structural differences are summarized below.

TABLE 1. Structural comparison of the Leech modular cone and the Guinand–Weil space.

Structural Property	Leech Modular Cone (Λ_{24})	Guinand–Weil Explicit Space (ζ)
Underlying Space	Discrete shells $m \in \mathbb{N}_{\geq 2}$ of $\Lambda_{24} \subset \mathbb{R}^{24}$	Continuous test functions $g \in C_c^\infty(\mathbb{R})$
Symmetry Group	$\text{Co}_0 = \text{Aut}(\Lambda_{24})$	Reflection $t \mapsto -t$ together with the Fourier-analytic structure of the explicit formula
Dimension Reduction	Harmonic averaging reduces the relevant conditions to $C(k) \approx k^2/48$	Prime-power evaluations form an infinite family without a corresponding finite group reduction
Spectral Boundary	Geometric exclusion zone: no vectors of squared norm 2, hence $S_1 = \emptyset$	No analogous hard lower spectral cut-off; zero-counting density grows with height
Duality Target	Annihilation: $\sum W_m v_k(m) = \mathbf{0}$	Positivity of a quadratic/test-function functional in the explicit formula

Proposition 5.1 (Structural Non-Implication). *The existence of non-negative annihilating measures on the finite-dimensional modular cone $\mathcal{C}_k(\Lambda_{24})$ does not, by itself, furnish an implication concerning the positive definiteness of the Guinand–Weil quadratic functional.*

Proof. Let $A_{\leq k}$ denote the cumulative Leech evaluation matrix. A positive vector $W > \mathbf{0}$ satisfying

$$A_{\leq k}W = \mathbf{0}$$

exists because the finite-dimensional cone generated by the relevant evaluation vectors contains the origin in its relative interior, as established in Part II. This property relies on the finite-dimensional modular quotient and the associated Co_0 -invariant reduction.

By contrast, the Guinand–Weil explicit formula involves an infinite family of prime-power evaluation functionals. Under a standard symmetric normalization, it has the schematic form

$$\mathcal{W}(f) = \int_{-\infty}^{\infty} f(x) \left(\frac{e^{x/2} + e^{-x/2}}{2 \cosh x} \right) dx - \sum_{p,m} \frac{\log p}{p^{m/2}} [f(\log p^m) + f(-\log p^m)], \quad (5.1)$$

with Fourier transform

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{ix\xi} dx.$$

The prime-power terms therefore constitute an infinite collection of point-evaluation functionals at the locations $\pm \log(p^m)$, with no direct analogue of the finite-dimensional Co_0 -invariant reduction.

Consequently, a positive annihilator in the finite-dimensional Leech quotient does not, without an additional theorem relating the two dual systems, constrain the positive definiteness of the unreduced Guinand–Weil functional. $\square$

Remark 5.2. The proposition is a statement about logical structure rather than a negative result concerning possible future connections. A genuine transfer would require an explicit map between the relevant dual spaces together with a proof that the positivity or annihilation property is preserved under that map.

6. CONCLUSION AND RESEARCH ROADMAP

Part III develops a conditional asymptotic framework for the infinite-dimensional limit of Leech lattice spherical cubatures.

- (1) We introduced the Archimedean Damping Kernel $\mathcal{K}_{s,\beta}(m)$ and showed that Gate 1 follows from the polynomial-exponential rescaled weight growth hypothesis.
- (2) We formulated the Spectral Horizon Conjecture

$$m^*(k) \asymp \frac{k^2}{96}.$$

The saddle-point analysis identifies the underlying k/m radial scaling, while the specific constant $1/96$ remains conjectural and is motivated by the numerical desert phenomenon and associated cone geometry.

- (3) We established weak-* compactness in the sequence space $\ell^1(\mathbb{N}_{\geq 2})$ and proved the Escape of Outer Mass theorem in $\sigma(\ell^1, c_0)$.
- (4) We identified structural differences between finite-dimensional modular cone duality on Λ_{24} and the infinite-dimensional Guinand–Weil positivity problem. In particular, the finite-dimensional annihilation mechanism does not by itself furnish a positivity theorem for the Guinand–Weil functional.

The principal open mathematical challenges are therefore:

- (1) proving Hypothesis 2.2 directly from the dual cone geometry or from the optimization/KKT structure;
- (2) deriving the precise horizon constant $1/96$, if the conjectured scaling is correct, from a rigorous modular saddle-point analysis;
- (3) establishing an exact normalization protocol for Gate 5 so that a non-trivial limiting functional can be identified; and
- (4) determining whether any rigorous correspondence exists between the finite-dimensional modular cone and the test-function spaces appearing in the Guinand–Weil explicit formula.

Thus the present work isolates the functional-analytic conditions required for a meaningful continuum limit without asserting that those conditions are automatically satisfied by the finite-dimensional constructions.

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