Phonon Dynamics, Cauchy–Lamé Reduction, and Design-Controlled Isotropy in the 24-Dimensional Leech Crystal

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Abstract

While the spectral geometry of isolated minimal shells in the Niemeier landscape was established in preceding works, the macroscopic elasticity and collective lattice dynamics of the infinite periodic Leech crystal $\Lambda_{24} \subset \mathbb{R}^{24}$ have remained open. In this paper, we formulate the Born–von Kármán dynamical matrix $D(\mathbf{k}) \in \mathrm{Mat}_{24}(\mathbb{C})$ across the 24-dimensional Brillouin zone for the class of admissible pairwise interactions $\mathcal{P}_{\mathrm{elast}}$ satisfying $\sum_{\mathbf{R}\neq\mathbf{0}} \left(\|\mathbf{R}\|^2|f'| + \|\mathbf{R}\|^4|f''|\right) < \infty$.

By Venkov's theorem (1984), every metric shell of the Leech lattice constitutes a spherical design of degree $t \geq 11$. We prove that this design property forces the shell-summed fourth moments entering the quadratic acoustic tensor to equal their spherical averages, giving the leading acoustic tensor the unique O(24)-invariant isotropic form $D^{(2)}_{ab}(\mathbf{k}) = (S_1+S_2)\|\mathbf{k}\|^2\delta_{ab} + 2S_2k_ak_b$, where $S_1 = -P_0$ is the radial virial stress and $S_2 = \frac{1}{312}\sum_{\mathbf{R}\neq\mathbf{0}} \|\mathbf{R}\|^4 f''(\|\mathbf{R}\|^2)$. Under the physical condition of mechanical zero-stress equilibrium ($S_1 = -P_0 = 0$), the microscopic central-force Cauchy symmetry implies $\lambda = \mu = S_2$, reducing the leading acoustic tensor to a single independent modulus. Consequently, the continuum acoustic sound velocities satisfy the exact algebraic ratio $v_L/v_T = \sqrt{3}$.

Furthermore, we establish the **Design-Controlled Long-Wavelength Isotropy Theorem**: for potentials in $\mathcal{P}^{(10)}$ whose lattice sums converge through degree 12 in $\|\mathbf{R}\|$, the 11-design property forces all relevant shell moments through degree 10 to equal their spherical averages, excluding all directional anisotropic contributions to $D(\mathbf{k})$ through order $\|\mathbf{k}\|^8$. The 23 transverse acoustic branches expand identically as $\omega_T(\mathbf{k}) = v_T\|\mathbf{k}\| + \alpha_3\|\mathbf{k}\|^3 + \alpha_5\|\mathbf{k}\|^5 + \alpha_7\|\mathbf{k}\|^7 + O(\|\mathbf{k}\|^9)$, with explicit closed-form rational coefficients α_j. Order $\|\mathbf{k}\|^{10}$ in $D(\mathbf{k})$ (order $\|\mathbf{k}\|^9$ in frequency) is the first order at which the 11-design hypothesis alone no longer excludes polarization splitting. Finally, we provide an algebraic formalization in Lean 4 and a numerical verification script in Python.

Contents

1 Introduction

The classification of 24-dimensional positive-definite even unimodular lattices [7, 4] and the universal optimality of the Leech lattice Λ_{24} proven by Cohn, Kumar, Miller, Radchenko, and Viazovska [3] establish that Λ_{24} minimizes potential energy among all point configurations in $\mathbb{R}^{24}$ of unit density for every completely monotonic function of squared Euclidean distance.

In preceding companion works on Niemeier and Leech shell configurations:

1. **Siegel Modular Rigidities [8]:** Pairwise separation of the twenty-four Niemeier lattices was shown to occur strictly at genus $g = 4$ through ordered orthogonal 4-frames of roots $a(I_4)$.
2. **Leech Minimal Shell Statics [9]:** The 4,520,880-dimensional Riemannian Hessian of the isolated minimal shell on $(S^{23})^{196560}$ under chordal potential $V(u) = u^{-2}$ was solved via Co_0-commutant reduction to $\mathbb{Q}^{12}$, establishing the rational acoustic ground state $\lambda_{\text{ground}} = \frac{73073}{58982400}$ and screening ratio $\kappa = \frac{730}{3597}$.
3. **Root Shell Stability Dichotomy [10]:** The 23 rooted Niemeier minimal shells were shown to exhibit a universal instability dichotomy under $V(u) = u^{-2}$, wherein only $A_1^{\oplus 24}$ achieves dynamic stability ($H \succeq 0$, $\lambda_S = \frac{49}{128}$) while the remaining 22 lattices collapse due to roots at squared distance 2.

Despite these advances, the macroscopic elasticity and collective lattice dynamics of the **infinite periodic Leech crystal** $\Lambda_{24} \subset \mathbb{R}^{24}$ have remained open. The transition from an isolated spherical shell $(S^{23})^{196560}$ to the infinite Bravais lattice collapses the dynamical problem from a 4.52-million-dimensional operator to a continuous family of 24×24 Hermitian matrices $D(\mathbf{k})$ parameterized by quasi-momentum $\mathbf{k} \in \mathbb{R}^{24}/\Lambda_{\text{rec}}$.

1.1 Convergence Conditions and Potential Classes

A fundamental mathematical distinction separates finite spherical shells from infinite periodic lattices. In dimension $d = 24$, the number of lattice points within a spherical shell of radius r grows asymptotically as $r^{23}dr$. For an inverse-power central pair potential $V(r) = r^{-2s}$, the local force constants decay as $\nabla^2 V(r) \sim r^{-2s-2}$. Consequently, the force-constant lattice sum:

$$\sum_{\mathbf{R}\in\Lambda_{24}\setminus\{\mathbf{0}\}} |\nabla_a \nabla_b V(\mathbf{R})| \sim \int^{\infty} r^{23} r^{-2s-2}\, dr = \int^{\infty} r^{21-2s}\, dr \tag{1}$$

converges absolutely if and only if $2s - 21 > 1 \iff s > 11$ (or $p > 22$ for $V(r) = r^{-p}$).

However, the leading quadratic acoustic coefficient contains an additional factor of $\|\mathbf{R}\|^2$:

$$\sum_{\mathbf{R}\in\Lambda_{24}\setminus\{\mathbf{0}\}} \|\mathbf{R}\|^2 |\nabla_a \nabla_b V(\mathbf{R})| \sim \int^{\infty} r^{23} r^{-2s}\, dr = \int^{\infty} r^{23-2s}\, dr, \tag{2}$$

which converges absolutely if and only if $2s - 23 > 1 \iff s > 12$ (or $p > 24$). The value $s = 12$ represents a borderline logarithmic divergence for acoustic elastic constants.

Therefore, standard Born–von Kármán lattice dynamics is well-defined only within specific analytical frameworks:

1. **Continuum Elasticity Class ($\mathcal{P}_{\text{elast}}$):** Smooth pair potentials $V(r) = f(r^2) \in C^2((0,\infty))$ satisfying:
$$\sum_{\mathbf{R}\in\Lambda_{24}\setminus\{\mathbf{0}\}} \left(\|\mathbf{R}\|^2 |f'(\|\mathbf{R}\|^2)| + \|\mathbf{R}\|^4 |f''(\|\mathbf{R}\|^2)|\right) < \infty. \tag{3}$$
This condition guarantees absolute convergence of the leading acoustic tensor C_{acbd}.

Lattice Dynamical Quantity	Radial Asymptotic Integral	Absolute Convergence
Force-constant sum $\sum_{\mathbf{R}} \lvert\nabla^2 V(\mathbf{R})\rvert$	$\int^\infty r^{21-2s}\,dr$	$\mathbf{s > 11}$ $(p > 22)$
Acoustic tensor $\sum_{\mathbf{R}} \lVert\mathbf{R}\rVert^2 \lvert\nabla^2 V(\mathbf{R})\rvert$	$\int^\infty r^{23-2s}\,dr$	$\mathbf{s > 12}$ $(p > 24)$
Dynamical matrix $D(\mathbf{k})$ (via $\lvert 1-\cos(\mathbf{k}\cdot\mathbf{R})\rvert \le \frac{1}{2}\lVert\mathbf{k}\rVert^2\lVert\mathbf{R}\rVert^2$)	$\int^\infty r^{23-2s}\,dr$	$\mathbf{s > 12}$ $(p > 24)$
Higher Taylor coefficient $D^{(2j)}(\mathbf{k})$ $(j \le 5)$	$\int^\infty r^{23+2j-2s}\,dr$	$\mathbf{s > 12 + j}$

Table 1: Exact convergence thresholds for power-law interactions $V(r) = r^{-2s} = r^{-p}$ in dimension $d = 24$.

2. **Higher-Order Dispersion Class ($\mathcal{P}^{(10)}$):** Smooth potentials satisfying the sufficient decay condition:

$$\sum_{\mathbf{R}\in\Lambda_{24}\setminus\{\mathbf{0}\}} \sum_{j=1}^{5} \left(\lVert\mathbf{R}\rVert^{2j}\lvert f'(\lVert\mathbf{R}\rVert^2)\rvert + \lVert\mathbf{R}\rVert^{2j+2}\lvert f''(\lVert\mathbf{R}\rVert^2)\rvert\right) < \infty. \tag{4}$$

   This condition guarantees that dominated convergence applies to the Taylor expansion of $D(\mathbf{k})$ through order $\lVert\mathbf{k}\rVert^{10}$.

3. **Renormalized Long-Range Potentials ($s \le 12$):** For interactions where direct sums diverge (such as $V(r) = r^{-4}$ where $s = 2$), regularized force constants require background neutralization and analytic continuation of the Leech Epstein zeta function.

1.2 Summary of Main Results

This paper establishes the collective lattice dynamics and continuum elasticity of Λ_{24}:

1. **Theorem A (Shell Moment Isotropy, Theorem 4.1):** For any potential $f \in \mathcal{P}_{\mathrm{elast}}$, the spherical 11-design property of every Leech shell forces the shell-summed fourth moments entering the quadratic acoustic tensor to equal their spherical averages, giving $D^{(2)}_{ab}(\mathbf{k}) = (S_1 + S_2)\lVert\mathbf{k}\rVert^2\delta_{ab} + 2S_2 k_a k_b$.

2. **Theorem B (Zero-Stress Cauchy Reduction, Theorem 4.5):** At mechanical zero-stress equilibrium ($S_1 = -P_0 = 0$), the microscopic central-force Cauchy symmetry implies $\lambda = \mu = S_2 = \frac{1}{312}\sum_{\mathbf{R}\neq\mathbf{0}} \lVert\mathbf{R}\rVert^4 f''(\lVert\mathbf{R}\rVert^2)$, establishing that the macroscopic elastic response is governed by a **single independent modulus** μ.

3. **Corollary C (Conditional Zero-Stress Sound Velocity Ratio, Theorem 4.6):** At zero-stress equilibrium, the longitudinal (v_L) and transverse (v_T) sound speeds satisfy:

$$\frac{v_L}{v_T} = \sqrt{3} \approx 1.7320508\ldots, \tag{5}$$

   which is an exact, dimension-independent algebraic invariant.

4. **Theorem D (Design-Controlled Long-Wavelength Isotropy, Theorem 5.1):** For potentials in $\mathcal{P}^{(10)}$, the 11-design property forces all shell moments through degree 10 to equal their spherical averages, excluding all directional anisotropic contributions to $D(\mathbf{k})$ through order $\lVert\mathbf{k}\rVert^8$. The 23 transverse acoustic branches expand identically as $\omega_T(\mathbf{k}) = v_T\lVert\mathbf{k}\rVert + \alpha_3\lVert\mathbf{k}\rVert^3 + \alpha_5\lVert\mathbf{k}\rVert^5 + \alpha_7\lVert\mathbf{k}\rVert^7 + O(\lVert\mathbf{k}\rVert^9)$. Order $\lVert\mathbf{k}\rVert^{10}$ in $D(\mathbf{k})$ (order $\lVert\mathbf{k}\rVert^9$ in frequency) is the first order where design symmetry alone no longer excludes polarization splitting.

5. **Formal and Numerical Verification:** We provide an algebraic verification artifact in Lean 4 and a Python script evaluating the reduced continuum acoustic tensor.

2 Leech Lattice Geometry and Venkov's 11-Design Theorem

Let $\Lambda_{24} \subset \mathbb{R}^{24}$ denote the standard even unimodular Leech lattice normalized such that the minimal non-zero squared Euclidean norm is:

$$\min_{\mathbf{R}\in\Lambda_{24}\setminus\{\mathbf{0}\}} \|\mathbf{R}\|^2 = 4. \tag{6}$$

Because $\det(\Lambda_{24}) = 1$, the volume of the fundamental unit cell is $v_c = 1$. We set the particle mass to $M = 1$; since $v_c = 1$, the equilibrium mass density is identically $\rho = M/v_c = 1$.

The dual lattice is $\Lambda_{24}^* := \{\mathbf{y} \in \mathbb{R}^{24} : \langle \mathbf{x}, \mathbf{y}\rangle \in \mathbb{Z}\} = \Lambda_{24}$. The reciprocal lattice preserving plane-wave phases $e^{i\mathbf{k}\cdot\mathbf{R}}$ is:

$$\Lambda_{\text{rec}} = 2\pi\Lambda_{24}^* = 2\pi\Lambda_{24}. \tag{7}$$

The first Brillouin zone is the Voronoi cell $\mathcal{B}_{24} := \mathbb{R}^{24}/\Lambda_{\text{rec}} = \mathbb{R}^{24}/(2\pi\Lambda_{24})$.

With $q := e^{2\pi i\tau}$, the theta series of Λ_{24} is:

$$\Theta_{\Lambda_{24}}(q) = 1 + \sum_{n=1}^{\infty} N(2n)q^n = 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + \ldots, \tag{8}$$

where $n = \|\mathbf{R}\|^2/2$ denotes the half-squared norm. The coefficient of q^1 vanishes identically ($N(2) = 0$) because Λ_{24} contains no roots.

Theorem 2.1 (Venkov's Spherical 11-Design Theorem [11, 6])**.** *Every non-empty metric shell $S_n = \{\mathbf{R} \in \Lambda_{24} : \|\mathbf{R}\|^2 = 2n\}$ $(n \geq 2)$ of the Leech lattice forms an antipodal spherical design of degree $t = 11$ on $S^{23}_{\sqrt{2n}}$.*

Conceptual Overview of the Modular-Form Mechanism [11, 6]. Let $P(\mathbf{x})$ be a non-zero harmonic polynomial on $\mathbb{R}^{24}$ of homogeneous degree $k \geq 1$. The weighted theta series $\Theta_{\Lambda_{24},P}(\tau) := \sum_{\mathbf{R}\in\Lambda_{24}} P(\mathbf{R})q^{\|\mathbf{R}\|^2/2}$ is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $w = 12 + k$. Because $k \geq 1$, $P(\mathbf{0}) = 0$, so $\Theta_{\Lambda_{24},P}$ is a cusp form: $\Theta_{\Lambda_{24},P} \in S_{12+k}(\mathrm{SL}_2(\mathbb{Z}))$.

For odd k, $P(-\mathbf{R}) = -P(\mathbf{R})$ forces $\Theta_{\Lambda_{24},P} \equiv 0$ by antipodal symmetry. For even $k \in \{2, 4, 6, 8, 10\}$, the dimensions of $S_{12+k}(\mathrm{SL}_2(\mathbb{Z}))$ are $\dim S_{14} = 0$ and $\dim S_{16} = \dim S_{18} = \dim S_{20} = \dim S_{22} = 1$. In each 1-dimensional space, the unique cusp form has a non-zero q^1 coefficient ($a_1 \neq 0$). However, because Λ_{24} contains no vectors of squared norm 2 ($N(2) = 0$), the q^1 coefficient of $\Theta_{\Lambda_{24},P}$ must vanish, forcing $\Theta_{\Lambda_{24},P}(\tau) \equiv 0$ identically for all $1 \leq k \leq 11$. Consequently, on every shell S_n, $\sum_{\mathbf{R}\in S_n} P(\mathbf{R}) = 0$ for all harmonic polynomials of degree $1 \leq k \leq 11$. □

Corollary 2.2 (Exact Shell Moments Through Degree 4)**.** *On every shell S_n with norm $r_n^2 = 2n$ and multiplicity N_n:*

$$\sum_{\mathbf{R}\in S_n} R_c R_d = \frac{N_n r_n^2}{24}\delta_{cd}, \tag{9}$$

$$\sum_{\mathbf{R}\in S_n} R_a R_b R_c R_d = \frac{N_n r_n^4}{24 \times 26}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}). \tag{10}$$

Proof. For any spherical design of degree $t \geq 4$ on S_R^{d-1} in dimension $d = 24$, the fourth-rank polynomial moment must be an $\mathrm{O}(d)$-invariant isotropic tensor:

$$\int_{S_R^{d-1}} x_a x_b x_c x_d \, d\sigma(\mathbf{x}) = C_4 \left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right). \tag{11}$$

Taking the double trace by contracting with $\delta_{ab}\delta_{cd}$ yields:

$$\int_{S_R^{d-1}} \|\mathbf{x}\|^4 \, d\sigma(\mathbf{x}) = R^4 = C_4\big(d^2 + 2d\big) = C_4 \, d(d+2) \implies C_4 = \frac{R^4}{d(d+2)}. \tag{12}$$

Substituting $d = 24$ gives $d(d+2) = 24 \times 26 = 624$, establishing the rational denominator $1/624$ in Eq. (10). $\square$

3 The Born–von Kármán Dynamical Matrix

Consider an infinite Bravais crystal with one particle per unit cell, of mass density $\rho = M/v_c = 1$. The particles interact via an admissible pairwise potential $V(r) = f(r^2) \in \mathcal{P}_{\text{elast}}$.

Let $\mathbf{u}(\mathbf{R})$ denote the displacement of the particle at equilibrium lattice site $\mathbf{R} \in \Lambda_{24}$. The harmonic potential energy is:

$$E^{(2)} = \frac{1}{4} \sum_{\mathbf{R} \neq \mathbf{R}'} \sum_{a,b=1}^{24} \Phi_{ab}(\mathbf{R} - \mathbf{R}')\big(u_a(\mathbf{R}) - u_a(\mathbf{R}')\big)\big(u_b(\mathbf{R}) - u_b(\mathbf{R}')\big), \tag{13}$$

where the local force-constant tensor is:

$$\Phi_{ab}(\mathbf{R}) := \left.\frac{\partial^2 f(\|\mathbf{x}\|^2)}{\partial x_a \partial x_b}\right|_{\mathbf{x}=\mathbf{R}} = 2\delta_{ab} f'(\|\mathbf{R}\|^2) + 4R_a R_b f''(\|\mathbf{R}\|^2). \tag{14}$$

By Bloch's theorem, plane-wave displacements $\mathbf{u}(\mathbf{R}) = \mathbf{e}e^{i(\mathbf{k}\cdot\mathbf{R} - \omega t)}$ diagonalize the equations of motion $\rho\ddot{\mathbf{u}} = -\nabla_{\mathbf{u}} E^{(2)}$, yielding the secular equation:

$$\rho\omega^2(\mathbf{k}) e_a(\mathbf{k}) = \sum_{b=1}^{24} D_{ab}(\mathbf{k}) e_b(\mathbf{k}), \tag{15}$$

where the Born–von Kármán dynamical matrix $D(\mathbf{k}) \in \mathrm{Mat}_{24}(\mathbb{C})$ is:

$$\boxed{D_{ab}(\mathbf{k}) := \sum_{\mathbf{R} \in \Lambda_{24} \setminus \{\mathbf{0}\}} \big(1 - \cos(\mathbf{k}\cdot\mathbf{R})\big) \left[2\delta_{ab} f'(\|\mathbf{R}\|^2) + 4R_a R_b f''(\|\mathbf{R}\|^2)\right].} \tag{16}$$

Proposition 3.1 (Properties of $D(\mathbf{k})$ for $f \in \mathcal{P}_{\text{elast}}$)**.** *The dynamical matrix satisfies:*

1. ***Uniform Convergence:*** *Because $f \in \mathcal{P}_{elast}$ and $|1 - \cos(\mathbf{k}\cdot\mathbf{R})| \le \frac{1}{2}(\mathbf{k}\cdot\mathbf{R})^2 \le \frac{1}{2}\|\mathbf{k}\|^2\|\mathbf{R}\|^2$, the sum in Eq. (16) converges absolutely and uniformly for all $\mathbf{k}$ on compact subsets of $\mathbb{R}^{24}$.*

2. ***Hermiticity and Reality:*** *$D_{ab}(\mathbf{k}) = D_{ba}(\mathbf{k}) \in \mathbb{R}$ for all $\mathbf{k}$ by lattice inversion symmetry $\mathbf{R} \in \Lambda_{24} \iff -\mathbf{R} \in \Lambda_{24}$.*

3. ***Acoustic Zero Modes:*** *$D(\mathbf{0}) \equiv \mathbf{0}_{24\times 24}$, guaranteeing that all 24 acoustic branches vanish at the zone center ($\omega_j(\mathbf{0}) = 0$).*

4. ***Periodicity:*** *$D(\mathbf{k} + 2\pi\mathbf{G}) = D(\mathbf{k})$ for all reciprocal vectors $2\pi\mathbf{G} \in 2\pi\Lambda_{24}$.*

4 Continuum Elasticity, Prestress, and Conditional Cauchy–Lamé Reduction

4.1 Long-Wavelength Acoustic Expansion

Expanding $1 - \cos(\mathbf{k}\cdot\mathbf{R}) = \frac{1}{2}(\mathbf{k}\cdot\mathbf{R})^2 - \frac{1}{24}(\mathbf{k}\cdot\mathbf{R})^4 + \dots$ in Eq. (16) yields:

$$D_{ab}(\mathbf{k}) = \sum_{c,d=1}^{24} C_{acbd} k_c k_d + O(\|\mathbf{k}\|^4), \tag{17}$$

where the continuum acoustic tensor C_{acbd} is:

$$C_{acbd} = \sum_{\mathbf{R}\in\Lambda_{24}\setminus\{\mathbf{0}\}} R_c R_d \left[\delta_{ab} f'(\|\mathbf{R}\|^2) + 2R_a R_b f''(\|\mathbf{R}\|^2)\right]. \tag{18}$$

4.2 Design-Forced Continuum Isotropy

Theorem 4.1 (Design-Forced Continuum Isotropy)**.** *For any potential $f \in \mathcal{P}_{elast}$, the shell-summed fourth moments entering the quadratic acoustic tensor coincide with their spherical averages, forcing $D^{(2)}_{ab}(\mathbf{k})$ to possess the unique* O(24)*-invariant isotropic form:*

$$\boxed{D^{(2)}_{ab}(\mathbf{k}) = (S_1 + S_2)\|\mathbf{k}\|^2 \delta_{ab} + 2S_2 k_a k_b,} \tag{19}$$

where S_1 and S_2 are the convergent shell sums:

$$S_1 := \frac{1}{24} \sum_{\mathbf{R}\in\Lambda_{24}\setminus\{\mathbf{0}\}} \|\mathbf{R}\|^2 f'(\|\mathbf{R}\|^2) = \frac{1}{24}\sum_{n=2}^{\infty} N_n r_n^2 f'(2n), \tag{20}$$

$$S_2 := \frac{1}{312} \sum_{\mathbf{R}\in\Lambda_{24}\setminus\{\mathbf{0}\}} \|\mathbf{R}\|^4 f''(\|\mathbf{R}\|^2) = \frac{1}{312}\sum_{n=2}^{\infty} N_n r_n^4 f''(2n). \tag{21}$$

Proof. Decomposing the lattice sum in Eq. (18) shell-by-shell:

$$C_{acbd} = \sum_{n=2}^{\infty}\left[\delta_{ab} f'(2n) \sum_{\mathbf{R}\in S_n} R_c R_d + 2f''(2n) \sum_{\mathbf{R}\in S_n} R_a R_b R_c R_d\right]. \tag{22}$$

By Corollary 2.2, substituting the 2nd and 4th moment identities yields:

$$\begin{aligned} C_{acbd} &= \sum_{n=2}^{\infty}\left[\delta_{ab} f'(2n)\frac{N_n r_n^2}{24}\delta_{cd} + 2f''(2n)\frac{N_n r_n^4}{24\times 26}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\right] \\ &= S_1\delta_{ab}\delta_{cd} + S_2(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}). \end{aligned} \tag{23}$$

Contracting with $k_c k_d$:

$$\sum_{c,d=1}^{24} C_{acbd} k_c k_d = S_1\|\mathbf{k}\|^2\delta_{ab} + S_2\left(\|\mathbf{k}\|^2\delta_{ab} + k_a k_b + k_a k_b\right) = (S_1+S_2)\|\mathbf{k}\|^2\delta_{ab} + 2S_2 k_a k_b, \tag{24}$$

which proves Eq. (19). ☐

4.3 Spectral Structure and Acoustic Stability

The isotropic form of the acoustic tensor has an immediate spectral consequence. Let $\mathbf{k} \neq \mathbf{0}$ and write $\widehat{\mathbf{k}} := \frac{\mathbf{k}}{\|\mathbf{k}\|}$. Then $\mathbb{R}^{24} = \mathrm{span}\{\widehat{\mathbf{k}}\} \oplus \widehat{\mathbf{k}}^{\perp}$, with $\dim \widehat{\mathbf{k}}^{\perp} = 23$.

Proposition 4.2 (Exact Transverse Degeneracy of the Continuum Tensor)**.** *Suppose that the acoustic tensor has the isotropic form of Eq. (19). Then:*

1. *Every transverse vector $\mathbf{v} \in \mathbf{k}^{\perp}$ is an eigenvector of $D^{(2)}(\mathbf{k})$ with eigenvalue:*

$$\lambda_T(\mathbf{k}) = (S_1 + S_2)\|\mathbf{k}\|^2. \tag{25}$$

2. *The longitudinal vector $\mathbf{k}$ is an eigenvector with eigenvalue:*

$$\lambda_L(\mathbf{k}) = (S_1 + 3S_2)\|\mathbf{k}\|^2. \tag{26}$$

3. *Consequently, the transverse eigenspace has exact multiplicity* 23*, and the longitudinal eigenspace has multiplicity* 1*.*

Proof. If $\mathbf{v} \in \mathbf{k}^\perp$, then $\mathbf{k} \cdot \mathbf{v} = 0$. Therefore:

$$D^{(2)}(\mathbf{k})\mathbf{v} = (S_1 + S_2)\|\mathbf{k}\|^2\mathbf{v} + 2S_2\mathbf{k}(\mathbf{k} \cdot \mathbf{v}) = (S_1 + S_2)\|\mathbf{k}\|^2\mathbf{v}. \tag{27}$$

For the longitudinal direction:

$$D^{(2)}(\mathbf{k})\mathbf{k} = (S_1 + S_2)\|\mathbf{k}\|^2\mathbf{k} + 2S_2\mathbf{k}(\mathbf{k} \cdot \mathbf{k}) = (S_1 + 3S_2)\|\mathbf{k}\|^2\mathbf{k}. \tag{28}$$

Since $\dim \mathbf{k}^\perp = 23$ in dimension 24, the claimed multiplicities follow. □

Proposition 4.3 (Continuum Acoustic Stability Conditions)**.** *For every $\mathbf{k} \neq \mathbf{0}$, the quadratic acoustic tensor $D^{(2)}(\mathbf{k})$ is positive semidefinite if and only if:*

$$S_1 + S_2 \geq 0, \qquad S_1 + 3S_2 \geq 0. \tag{29}$$

It is positive definite on the non-zero acoustic subspace if and only if both inequalities are strict.

Proof. By Proposition 4.2, the complete spectrum of $D^{(2)}(\mathbf{k})$ consists of $(S_1 + S_2)\|\mathbf{k}\|^2$ with multiplicity 23 and $(S_1 + 3S_2)\|\mathbf{k}\|^2$ with multiplicity 1. Since $\|\mathbf{k}\|^2 > 0$, positive semidefiniteness is equivalent to both eigenvalues being non-negative. □

Corollary 4.4 (Zero-Stress Stability)**.** *At mechanical zero stress ($S_1 = 0$), the continuum acoustic tensor is positive semidefinite if and only if $S_2 \geq 0$, and positive definite away from the zone center if and only if $S_2 > 0$.*

4.4 Microscopic Elasticity, Prestress, and Cauchy Symmetry

In finite-strain continuum mechanics [1, 12], under homogeneous displacement gradients $\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij}$, the potential energy per unit volume expands as:

$$\frac{E(\varepsilon) - E(0)}{v_c} = -\sum_{i,j} \sigma^0_{ij}\varepsilon_{ij} + \frac{1}{2}\sum_{i,j,k,l} B_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + O(\varepsilon^3), \tag{30}$$

where $\sigma^0_{ij} = -P_0\delta_{ij}$ is the initial hydrostatic Cauchy stress, and B_{ijkl} is the microscopic Born elastic stiffness tensor:

$$B_{ijkl} := \frac{1}{v_c}\sum_{\mathbf{R}\neq\mathbf{0}} \left[\delta_{ik}R_jR_lf'(\|\mathbf{R}\|^2) + 2R_iR_jR_kR_lf''(\|\mathbf{R}\|^2)\right]. \tag{31}$$

Evaluating via Corollary 2.2:

$$B_{ijkl} = \delta_{ik}\delta_{jl}S_1 + S_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{32}$$

Theorem 4.5 (Conditional Cauchy–Lamé Reduction)**.** *Let $f \in \mathcal{P}_{elast}$ be an admissible central pairwise potential.*

1. ***Hydrostatic Prestress:*** *The radial virial stress is $P_0 = -S_1 = -\frac{1}{24}\sum_{\mathbf{R}\neq\mathbf{0}} \|\mathbf{R}\|^2 f'(\|\mathbf{R}\|^2)$.*

2. ***Acoustic Tensor with Prestress:*** *Matching the leading dynamical matrix $D^{(2)}(\mathbf{k})$ with the standard continuum form:*

$$D^{(2)}_{ab}(\mathbf{k}) = \mu\|\mathbf{k}\|^2\delta_{ab} + (\lambda + \mu)k_ak_b \tag{33}$$

gives the effective acoustic parameters:

$$\boxed{\mu = S_1 + S_2, \qquad \lambda = S_2 - S_1.} \tag{34}$$

Equivalently, expressing μ in terms of the stress-free shear modulus S_2 and virial pressure $P_0 = -S_1$ yields $\mu = S_2 - P_0$ and $\lambda + \mu = 2S_2$.

3. ***Zero-Stress Cauchy Reduction:*** *If the Leech crystal is at mechanical zero-stress equilibrium ($P_0 = -S_1 = 0$), then $B_{ijkl} = S_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ becomes completely symmetric under all permutations of (i, j, k, l). In particular, the Cauchy relation $B_{1122} = B_{1212}$ holds without prestress corrections:*

$$\boxed{\lambda = \mu = S_2 = \frac{1}{312} \sum_{\mathbf{R} \in \Lambda_{24} \setminus \{\mathbf{0}\}} \|\mathbf{R}\|^4 f''(\|\mathbf{R}\|^2).} \tag{35}$$

Theorem 4.6 (Conditional Zero-Stress Sound Velocity Ratio)**.** *Suppose the infinite Leech crystal under an admissible potential $f \in \mathcal{P}_{elast}$ is at mechanical zero-stress equilibrium ($P_0 = 0$), with positive shear modulus $\mu = S_2 > 0$ and mass density $\rho = 1$. Decomposing displacement space along wavevector $\mathbf{k}$ as $\mathbb{R}^{24} = \mathrm{span}\{\mathbf{k}\} \oplus \mathbf{k}^\perp$:*

- *The 23 transverse acoustic branches in $\mathbf{k}^\perp$ have speed: $v_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{S_2}$,*
- *The single longitudinal acoustic branch along $\mathbf{k}$ has speed: $v_L = \sqrt{\frac{\lambda+2\mu}{\rho}} = \sqrt{\frac{3\mu}{\rho}} = \sqrt{3S_2}$.*

Consequently, the sound velocity ratio is an exact algebraic constant:

$$\boxed{\frac{v_L}{v_T} = \sqrt{3} \approx 1.732050807568877\ldots} \tag{36}$$

Proof. For any isotropic acoustic tensor with parameters λ and μ in the absence of initial stress:

$$v_L^2 = \frac{\lambda + 2\mu}{\rho}, \qquad v_T^2 = \frac{\mu}{\rho}. \tag{37}$$

Substituting $\lambda = \mu = S_2$ from Theorem 4.5:

$$v_L^2 = \frac{\mu + 2\mu}{\rho} = \frac{3\mu}{\rho} \implies \frac{v_L^2}{v_T^2} = \frac{3\mu/\rho}{\mu/\rho} = 3 \implies \frac{v_L}{v_T} = \sqrt{3}. \tag{38}$$

□

5 Design-Controlled Long-Wavelength Isotropy

In typical crystals (such as cubic FCC or BCC lattices in $\mathbb{R}^3$), the acoustic dispersion curves $\omega(\mathbf{k})$ exhibit directional anisotropy at leading order $O(\|\mathbf{k}\|^3)$ due to cubic invariants $k_x^4 + k_y^4 + k_z^4$. The Leech lattice suppresses directional splitting through high order:

Order in $D(\mathbf{k})$	f'-Part Max Degree	f''-Part Max Degree	Constrained by 11-Design?
$\|\mathbf{k}\|^2$	2	4	**Yes** ($4 \leq 11$)
$\|\mathbf{k}\|^4$	4	6	**Yes** ($6 \leq 11$)
$\|\mathbf{k}\|^6$	6	8	**Yes** ($8 \leq 11$)
$\|\mathbf{k}\|^8$	8	10	**Yes** ($10 \leq 11$)
$\|\mathbf{k}\|^{10}$	10	12	**No** ($12 > 11$)

Table 2: Polynomial degrees appearing in the Taylor expansion of $D(\mathbf{k})$ compared against the 11-design threshold.

Theorem 5.1 (Design-Controlled Long-Wavelength Isotropy Theorem)**.** *Let $D(\mathbf{k})$ be the Born–von Kármán dynamical matrix of Λ_{24} under a potential $f \in \mathcal{P}^{(10)}$.*

1. *In the Taylor expansion* $D(\mathbf{k}) = \sum_{j=1}^{\infty} D^{(2j)}(\mathbf{k})$*, all terms* $D^{(2j)}(\mathbf{k})$ *for* $j \in \{1, 2, 3, 4\}$ *are strictly* $\mathrm{O}(24)$*-isotropic:*

$$D_{ab}(\mathbf{k}) = D_{ab}^{(2)}(\mathbf{k}) + D_{ab}^{(4)}(\mathbf{k}) + D_{ab}^{(6)}(\mathbf{k}) + D_{ab}^{(8)}(\mathbf{k}) + O(\|\mathbf{k}\|^{10}), \tag{39}$$

where each $D^{(2j)}$ *contains only* $\mathrm{O}(24)$*-invariant contractions:*

$$D_{ab}^{(2j)}(\mathbf{k}) = A_j \|\mathbf{k}\|^{2j} \delta_{ab} + B_j \|\mathbf{k}\|^{2j-2} k_a k_b. \tag{40}$$

2. *Order-by-order near* $\mathbf{k} = \mathbf{0}$*, the 23 transverse acoustic branches in* $\mathbf{k}^{\perp}$ *expand identically through 7th order in frequency:*

$$\boxed{\omega_{T,1}(\mathbf{k}) = \cdots = \omega_{T,23}(\mathbf{k}) = v_T \|\mathbf{k}\| + \alpha_3 \|\mathbf{k}\|^3 + \alpha_5 \|\mathbf{k}\|^5 + \alpha_7 \|\mathbf{k}\|^7 + O(\|\mathbf{k}\|^9).} \tag{41}$$

3. *The 11-design property forces all relevant shell moments through degree 10 to equal their spherical averages, excluding directional anisotropic contributions through order* $\|\mathbf{k}\|^8$ *in* $D(\mathbf{k})$*. Order* $\|\mathbf{k}\|^{10}$ *in* $D(\mathbf{k})$ *(order* $\|\mathbf{k}\|^9$ *in frequency) is the first order at which the 11-design hypothesis alone no longer excludes polarization splitting.*

Proof. Expanding the displacement factor in Eq. (16):

$$1 - \cos(\mathbf{k} \cdot \mathbf{R}) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j)!} (\mathbf{k} \cdot \mathbf{R})^{2j}. \tag{42}$$

Multiplying by $\Phi_{ab}(\mathbf{R}) = 2\delta_{ab} f'(R^2) + 4R_a R_b f''(R^2)$, the $2j$-th term is:

$$D_{ab}^{(2j)}(\mathbf{k}) = \frac{(-1)^{j-1}}{(2j)!} \sum_{\mathbf{R} \neq \mathbf{0}} (\mathbf{k} \cdot \mathbf{R})^{2j} \left[2\delta_{ab} f'(R^2) + 4R_a R_b f''(R^2) \right]. \tag{43}$$

To determine the tensorial structure, contract $D_{ab}^{(2j)}$ with an arbitrary test vector $\mathbf{v} \in \mathbb{R}^{24}$:

$$\sum_{a,b=1}^{24} v_a v_b D_{ab}^{(2j)}(\mathbf{k}) = \frac{(-1)^{j-1}}{(2j)!} \sum_{\mathbf{R} \neq \mathbf{0}} (\mathbf{k} \cdot \mathbf{R})^{2j} \left[2\|\mathbf{v}\|^2 f'(R^2) + 4(\mathbf{v} \cdot \mathbf{R})^2 f''(R^2) \right]. \tag{44}$$

The summand consists of homogeneous polynomials in $\mathbf{R}$ of degree $2j$ (first term) and degree $2j + 2$ (second term).

By Theorem 2.1 (Venkov 1984), every shell S_n of Λ_{24} is an 11-design. Therefore, every harmonic polynomial of degree $p \leq 11$ integrates to zero across every shell.

For $j \in \{1, 2, 3, 4\}$, the maximum polynomial degree is $p_{\max} = 2j + 2 \leq 2(4) + 2 = 10 \leq 11$ (Table 2). Because $f \in \mathcal{P}^{(10)}$, dominated convergence applies, and all harmonic polynomial components of degrees $2 \leq p \leq 10$ vanish identically upon shell summation. The only non-vanishing components are the spherically invariant trace parts. By the polarization identity for symmetric multilinear forms, controlling the scalar form for all $\mathbf{v}$ forces the tensor $D_{ab}^{(2j)}(\mathbf{k})$ to be strictly $\mathrm{O}(24)$-isotropic for all $j \leq 4$, establishing the form $A_j \|\mathbf{k}\|^{2j} \delta_{ab} + B_j \|\mathbf{k}\|^{2j-2} k_a k_b$.

At order $\|\mathbf{k}\|^{10}$ ($j = 5$), the f''-contribution involves degree-12 shell moments ($2j + 2 = 12$). Such moments are not constrained by the 11-design property alone. Therefore, the 11-design hypothesis guarantees isotropy through order $\|\mathbf{k}\|^8$, but by itself does not determine whether anisotropic degree-12 contributions vanish at order $\|\mathbf{k}\|^{10}$. $\square$

5.1 Explicit Dispersion Expansion Coefficients

Let $q := \|\mathbf{k}\|$. Under the hypotheses of Theorem 5.1, the transverse acoustic eigenvalue on $\mathbf{k}^\perp$ expands as:

$$\omega_T^2(\mathbf{k}) = a_2 q^2 + a_4 q^4 + a_6 q^6 + a_8 q^8 + O(q^{10}), \tag{45}$$

where $a_{2j} = A_j$ are the isotropic diagonal coefficients of $D^{(2j)}$. Assuming $a_2 = v_T^2 > 0$, the transverse frequency is obtained by taking the positive square root:

$$\omega_T(\mathbf{k}) = q\sqrt{a_2 + a_4 q^2 + a_6 q^4 + a_8 q^6 + O(q^8)} = v_T q + \alpha_3 q^3 + \alpha_5 q^5 + \alpha_7 q^7 + O(q^9). \tag{46}$$

A Taylor expansion of the square root directly yields the explicit rational expressions:

$$v_T = \sqrt{a_2}, \tag{47}$$

$$\alpha_3 = \frac{a_4}{2\sqrt{a_2}}, \tag{48}$$

$$\alpha_5 = \frac{a_6}{2\sqrt{a_2}} - \frac{a_4^2}{8a_2^{3/2}}, \tag{49}$$

$$\alpha_7 = \frac{a_8}{2\sqrt{a_2}} - \frac{a_4 a_6}{4a_2^{3/2}} + \frac{a_4^3}{16a_2^{5/2}}. \tag{50}$$

Because all 23 transverse branches share the exact same coefficients A_j, Eqs. (47)–(50) are identical across the entire 23-dimensional transverse subspace $\mathbf{k}^\perp$.

5.2 First Unconstrained Anisotropic Order

At order q^{10}, the f'' contribution has the form:

$$D_{ab}^{(10)}(\mathbf{k}) \supset \frac{(-1)^4}{10!} 4 \sum_{\mathbf{R}\neq\mathbf{0}} (\mathbf{k}\cdot\mathbf{R})^{10} R_a R_b f''(\|\mathbf{R}\|^2). \tag{51}$$

The associated shell moment is of degree 12 in $\mathbf{R}$. Decomposing this rank-12 moment into O(24) irreducible harmonic components contains, in general, a harmonic degree-12 component that is not forced to vanish by an 11-design condition.

Proposition 5.2 (Order of First Possible Polarization Splitting)**.** *The 11-design hypothesis alone guarantees the absence of polarization splitting in the transverse acoustic eigenvalues through order q^8 in $D(\mathbf{k})$ (order q^7 in frequency). At order q^{10} in $D(\mathbf{k})$ (order q^9 in frequency), the 11-design hypothesis alone does not exclude an anisotropic transverse operator.*

6 Long-Range Interactions and Epstein-Zeta Regularization

Remark 6.1 (Divergence of Unregularized Riesz Lattice Sums)**.** For long-range inverse-power potentials $V(r) = r^{-2s}$ with $s \le 12$ (such as $V(r) = r^{-4}$ where $s = 2$), the unregularized acoustic lattice sum diverges as $\int^\infty r^{23-2s} dr = \int^\infty r^{17} dr$, as established in Table 1. Consequently, $V(r) = r^{-4}$ does not belong to $\mathcal{P}_{\mathrm{elast}}$ or $\mathcal{P}^{(10)}$.

In statistical mechanics and Coulomb gas theory, macroscopic stability for $s \le d/2$ requires introducing an inert uniform neutralizing background of opposite charge density $\rho_{\mathrm{bg}} = -1/v_c = -1$. In this neutralized jellium framework, the infinite lattice energy is formally defined through the analytic continuation of the Leech Epstein zeta function:

$$E_{\Lambda_{24}}(s) := \sum_{\mathbf{R}\in\Lambda_{24}\setminus\{\mathbf{0}\}} \|\mathbf{R}\|^{-2s}, \quad \mathrm{Re}(s) > 12. \tag{52}$$

By the Rankin–Selberg method, $E_{\Lambda_{24}}(s)$ extends to a meromorphic function on $\mathbb{C}$ with a single simple pole at $s = 12$ associated with the volume divergence. Subtracting the background charge cancels the pole at $s = 12$, defining regularized force constants via the regular value $E_{\Lambda_{24}}(2)$. In this paper, all dynamical theorems (Theorems 4.1, 4.5, 4.6, and 5.1) are proven rigorously for the convergent potential classes $\mathcal{P}_{\text{elast}}$ and $\mathcal{P}^{(10)}$, avoiding reliance on unregularized long-range sums.

7 Lean 4 Formalization of the Algebraic Cauchy–Lamé Implication

Listing 1 presents the machine-checked proof artifact in Lean 4 (‘LeechElasticity.lean‘). It formalizes the algebraic implication: given an arbitrary isotropic fourth-rank tensor $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, the Cauchy symmetry $C_{1122} = C_{1212}$ algebraically forces $\lambda = \mu$, which in turn implies $v_L/v_T = \sqrt{3}$ over $\mathbb{Q}$.

```
import Mathlib.Data.Rat.Basic
import Mathlib.Data.Real.Basic
import Mathlib.Tactic.Linarith
import Mathlib.Tactic.Ring

namespace LeechElasticity

/-- General isotropic elasticity tensor parameters (Lame coefficients) -/
structure IsotropicElasticity (   : Type*) where
  λ_param :
  μ_param :

/-- Component evaluation of the isotropic 4th-rank stiffness tensor -/
def C_tensor (params : IsotropicElasticity ℚ) (i j k l : Fin 24) : ℚ :=
  let δ (a b : Fin 24) : ℚ := if a = b then 1 else 0
  params.λ_param * (δ i j * δ k l) +
  params.μ_param * (δ i k * δ j l + δ i l * δ j k)

/-- The central-force Cauchy symmetry condition: C_{ikjl} = C_{ijkl} -/
def SatisfiesCauchy (params : IsotropicElasticity ℚ) : Prop :=
      i j k l : Fin 24, C_tensor params i k j l = C_tensor params i j k l

/-- THEOREM: For any isotropic medium, the Cauchy condition forces λ = μ identically -/
theorem cauchy_lame_reduction (params : IsotropicElasticity ℚ)
    (h : SatisfiesCauchy params) : params.λ_param = params.μ_param := by
  have h_eval := h 0 1 0 1
  unfold C_tensor at h_eval
  have h00 : (if (0 : Fin 24) = 0 then (1 : ℚ) else 0) = 1 := rfl
  have h01 : (if (0 : Fin 24) = 1 then (1 : ℚ) else 0) = 0 := rfl
  have h11 : (if (1 : Fin 24) = 1 then (1 : ℚ) else 0) = 1 := rfl
  have h10 : (if (1 : Fin 24) = 0 then (1 : ℚ) else 0) = 0 := rfl
  revert h_eval
  simp only [h00, h01, h11, h10]
  intro h_eq
  linarith

/-- COROLLARY: The longitudinal-to-transverse sound velocity squared ratio is strictly 3
      -/
theorem sound_velocity_squared_ratio (params : IsotropicElasticity ℚ)
    (h_cauchy : SatisfiesCauchy params) (h_pos : params.μ_param > 0) :
    let v_L_sq := params.λ_param + 2 * params.μ_param
    let v_T_sq := params.μ_param
    v_L_sq / v_T_sq = 3 := by
  have h_eq : params.λ_param = params.μ_param := cauchy_lame_reduction params h_cauchy
  dsimp
  rw [h_eq]
  have h_denom : params.μ_param ≠ 0 := by linarith
  calc
    (params.μ_param + 2 * params.μ_param) / params.μ_param
      = (3 * params.μ_param) / params.μ_param := by ring_nf
    _ = 3 := mul_div_cancel_right 3 h_denom

end LeechElasticity
```

Listing 1: Formal proof in Lean 4: `LeechElasticity.lean`.

8 Numerical Verification of the Reduced Acoustic Tensor in Python

Listing 2 provides the verification script 'leech$_l$$attice_d$$ynamics.py$'$.Itconstructsthereducedcontinuumacousticten$ are degenerate to machine precision, confirms the ratio $v_L/v_T = \sqrt{3}$, and shows that adding an anisotropic perturbation breaks the 23-fold degeneracy.

```
#!/usr/bin/env python3
import numpy as np

def verify_reduced_acoustic_tensor():
    d = 24
    # Zero-stress equilibrium condition: S1 = -P0 = 0
    S1 = 0.0
    # Macroscopic shear modulus S2 = mu > 0
    S2 = 1.25

    # Choose an arbitrary propagation direction
    np.random.seed(42)
    k_dir = np.random.randn(d)
    k_dir /= np.linalg.norm(k_dir)
    k_mag = 0.01
    k = k_mag * k_dir
    k2 = float(np.dot(k, k))

    # Analytical dynamical matrix: D_ab = (S1 + S2)|k|^2 delta_ab + 2 S2 k_a k_b
    D = (S1 + S2) * k2 * np.eye(d) + 2.0 * S2 * np.outer(k, k)

    evals = np.linalg.eigvalsh(D)
    ta_evals = evals[:23]
    la_eval = evals[23]

    expected_ta = (S1 + S2) * k2
    expected_la = (S1 + 3.0 * S2) * k2

    max_ta_spread = np.max(ta_evals) - np.min(ta_evals)
    la_error = abs(la_eval - expected_la)
    ratio_sq = la_eval / ta_evals[0]

    print(f"23 Transverse Modes Spread : {max_ta_spread:.2e} (Strictly degenerate)")
    print(f"Longitudinal Mode Error    : {la_error:.2e}")
    print(f"Computed Ratio v_L^2 / v_T^2: {ratio_sq:.12f} (Expected: 3.0)")
    assert max_ta_spread < 1e-15, "Transverse degeneracy violated."
    assert abs(ratio_sq - 3.0) < 1e-14, "Velocity ratio failed."

    # Demonstrate that an anisotropic perturbation breaks the 23-fold degeneracy
    eps = 0.05 * S2
    D_perturbed = D + eps * k2 * np.diag(k_dir**2)
    evals_perturbed = np.linalg.eigvalsh(D_perturbed)
    ta_spread_perturbed = np.max(evals_perturbed[:23]) - np.min(evals_perturbed[:23])
    relative_spread = ta_spread_perturbed / abs(expected_ta)
    print(f"Perturbed Transverse Spread: {ta_spread_perturbed:.2e} (Relative: {
    relative_spread:.2e})")
    assert relative_spread > 1e-4, "Anisotropic perturbation test failed."
    print(">> Continuum elasticity and Cauchy-Lame reduction successfully verified.")

if __name__ == "__main__":
    verify_reduced_acoustic_tensor()
```

Listing 2: Numerical verification script: `leech`$_l$$attice_d$$ynamics.py$.

9 Conclusion, Limitations, and Outlook

9.1 Summary of Logical Architecture

This paper has established the collective dynamics and macroscopic elasticity of the infinite 24-dimensional Leech crystal Λ_{24}, subject to the stated convergence hypotheses on the pair potential.

1. **Design-forced continuum isotropy.** The spherical 11-design property of every non-empty Leech shell implies that the second- and fourth-rank shell moments coincide with their spherical averages. Consequently, for every $f \in \mathcal{P}_{\text{elast}}$, the quadratic acoustic tensor has the isotropic form

$$D^{(2)}_{ab}(\mathbf{k}) = (S_1 + S_2)\|\mathbf{k}\|^2 \delta_{ab} + 2S_2 k_a k_b. \tag{53}$$

   Thus the leading long-wavelength elastic response contains no directional anisotropy.

2. **Prestress and Cauchy reduction.** The microscopic central-force structure gives the Cauchy symmetry of the Born elastic tensor. When the crystal is evaluated at mechanical zero-stress equilibrium, $P_0 = -S_1 = 0$, this symmetry reduces the two isotropic Lamé coefficients to

$$\lambda = \mu = S_2 = \frac{1}{312} \sum_{\mathbf{R} \in \Lambda_{24} \setminus \{\mathbf{0}\}} \|\mathbf{R}\|^4 f''(\|\mathbf{R}\|^2). \tag{54}$$

   The zero-stress condition is therefore essential to the stated one-modulus reduction; the prestressed and zero-stress elastic descriptions should not be conflated.

3. **Acoustic velocity ratio.** Assuming positive shear modulus $\mu = S_2 > 0$ and unit mass density, the longitudinal and transverse acoustic velocities are

$$v_L = \sqrt{3S_2}, \qquad v_T = \sqrt{S_2}, \tag{55}$$

   and hence

$$\boxed{\frac{v_L}{v_T} = \sqrt{3}.} \tag{56}$$

   This conclusion follows algebraically from isotropy, central-force Cauchy symmetry, and zero prestress, rather than from a numerical fit.

4. **Design-controlled long-wavelength isotropy.** For potentials satisfying the stronger convergence assumptions in $\mathcal{P}^{(10)}$, the Taylor coefficients of the dynamical matrix through order $\|\mathbf{k}\|^8$ involve shell moments of degree at most 10. These moments are controlled by the 11-design property. Therefore the dynamical matrix has the structure

$$D_{ab}(\mathbf{k}) = \sum_{j=1}^{4} \left(A_j \|\mathbf{k}\|^{2j} \delta_{ab} + B_j \|\mathbf{k}\|^{2j-2} k_a k_b \right) + O(\|\mathbf{k}\|^{10}). \tag{57}$$

   In particular, all 23 transverse polarizations remain degenerate through this order.

5. **First unconstrained tensorial order.** At order $\|\mathbf{k}\|^{10}$ in $D(\mathbf{k})$, the f'' contribution contains degree-12 shell moments. The 11-design hypothesis alone does not determine these moments. Consequently, degree 12 is the first moment degree at which additional lattice information is required to decide whether polarization splitting occurs.

6. **Convergent versus regularized long-range interactions.** The preceding dynamical statements are established for absolutely convergent potential classes. In particular, inverse-power interactions with insufficient decay cannot simply be inserted into the same lattice sums. Long-range models require a separately specified regularization, background prescription, or other renormalization scheme.

7. **Formal and numerical verification.** The Lean 4 artifact isolates the algebraic Cauchy–Lamé implication and verifies the resulting squared velocity ratio over $\mathbb{Q}$. The accompanying Python calculation independently evaluates the reduced acoustic tensor, verifies the 23-fold transverse degeneracy, and demonstrates numerically that an explicitly introduced anisotropic perturbation destroys that degeneracy.

9.2 Scope and Limitations

Several qualifications are important for interpreting the results:

1. The 11-design property is a statement about polynomial moments on individual metric shells. It does not imply that the entire finite-$\mathbf{k}$ dynamical matrix is exactly O(24)-invariant. The Brillouin zone and the reciprocal lattice retain the discrete symmetry of Λ_{24}. The isotropy proved here is a controlled long-wavelength statement whose order is determined by the largest polynomial moment to which the design hypothesis applies.

2. The equality $\lambda = \mu$ is conditional on the central-force Cauchy relation together with the zero-stress reduction used in this paper. For a prestressed crystal, the distinction between the Born stiffness tensor, the incremental elastic tensor, and the acoustic tensor must be retained. Accordingly, the sound-velocity ratio $\sqrt{3}$ should not be interpreted as a universal finite-pressure identity.

3. The higher-order isotropy result does not assert that the coefficient of the first potentially anisotropic term is non-zero. At order $\|\mathbf{k}\|^{10}$, the 11-design property ceases to be sufficient to exclude anisotropy, but the actual degree-12 shell moments may contain additional structure that could further suppress or constrain polarization splitting.

9.3 Outlook

The framework developed here suggests three natural directions for further study:

1. **Determination of degree-12 shell moments:** Computing the explicit degree-12 harmonic moment tensor of Λ_{24} will decide whether the first order not controlled by Venkov's theorem nevertheless exhibits additional isotropy forced by Co_0.

2. **Explicit evaluation of $\alpha_3, \alpha_5, \alpha_7$:** Computing the radial lattice sums for specific rapidly decaying potentials (such as Gaussians or short-range Morse potentials) will provide explicit dispersion curves for the Leech crystal.

3. **Non-central and many-body forces:** Extending from central pair potentials to three-body or angular potentials would break the microscopic Cauchy relation, allowing an independent determination of the macroscopic Lamé parameters while preserving design-controlled isotropy.

More broadly, the Leech lattice provides an unusually clean setting in which three distinct physical mechanisms:

$$\boxed{\text{spherical design} \quad + \quad \text{central-force symmetry} \quad + \quad \text{zero prestress}} \tag{58}$$

lead respectively to long-wavelength isotropy, Cauchy symmetry, and the one-modulus acoustic reduction. Keeping these mechanisms logically separate makes clear which conclusions follow from lattice geometry, which follow from the interaction model, and which require an equilibrium condition.

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