

Voronoi Facets, Modular Selection Rules, and High-Order Elastic Ultraisotropy of the 24-Dimensional Leech Crystal

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Abstract

We establish the complete geometric facet census, harmonic selection rules, and long-wavelength collective phonon dynamics of the 24-dimensional periodic Leech crystal $\Lambda_{24} \subset \mathbb{R}^{24}$ under central pair potentials. The Leech lattice is normalized such that the minimal non-zero squared norm is $\min_{\mathbf{R} \neq \mathbf{0}} \|\mathbf{R}\|^2 = 4$.

First, we resolve the Voronoi facet structure of Λ_{24} . Using the exact covering radius $R_{\text{cov}} = \sqrt{2}$, we prove that Voronoi-relevant vectors are strictly bounded by $\|\mathbf{v}\|^2 < 8$. Combining this bound with Voronoi's affine coset uniqueness criterion, we prove that the Voronoi polytope $\text{Vor}(\Lambda_{24})$ possesses **exactly** 16,969,680 **facets**—comprising 196,560 kissing facets from the minimal shell S_2 and 16,773,120 shallow-hole facets from the second shell S_3 . We distinguish this count from the astronomical facet count of the contact polytope $\text{conv}(S_2)$.

Second, we formulate the harmonic multipole hierarchy governing shell moments to infinity via the double-cusp modular isomorphism $S_{12+d}^0(\text{SL}_2(\mathbb{Z})) \cong M_{d-12}(\text{SL}_2(\mathbb{Z}))$. For degree $d = 14$, the modular obstruction $M_2(\text{SL}_2(\mathbb{Z})) = \{0\}$ enforces $S_{26}^0 = \{0\}$, proving that all degree-14 harmonic shell moments vanish identically shell-by-shell. For degree $d = 12$, the one-dimensionality of $S_{24}^0 = \mathbb{C}\Delta^2$ links all degree-12 moments to the Ramanujan autoconvolution $c_m(\Delta^2)$; we evaluate these coefficients through $m = 25$, demonstrating that the initial sign alternation breaks down at $m = 11, 15, 18$. Via Mellin transformation, we prove that the completed L -function satisfies $\Lambda(s) > 0$ for all $s \in \mathbb{R}$, establishing that $L(\Delta^2, s) > 0$ on $(0, \infty)$ and ruling out single power-law degree-12 anisotropy annihilators.

Third, in the long-wavelength Born–von Kármán expansion, the 11-design property guarantees exact $O(24)$ -isotropy through order $\|\mathbf{k}\|^8$. At quadratic order, microscopic central-force Cauchy symmetry at zero prestress forces $\lambda = \mu = S_2$, yielding an exact sound velocity ratio $v_L/v_T = \sqrt{3}$. At order $\|\mathbf{k}\|^{10}$, degree-12 anisotropy enters through f'' via the functional $\mathcal{A}_{12}[f''] = \sum c_m f''(2m)$. In the two-shell model $(S_2 \cup S_3)$, the curvature condition $f''(6) = \frac{1}{48}f''(4)$ eliminates order- $\|\mathbf{k}\|^{10}$ anisotropy tensorially.

Fourth, at order $\|\mathbf{k}\|^{12}$, degree 14 is absent, but degree 12 re-emerges in both the f' and f'' terms. Through an exact irreducible tensor decomposition, we prove that order $\|\mathbf{k}\|^{12}$ is governed by two independent modular functionals: a scalar shift $\mathcal{F}_1[f'] = \sum c_m f'(2m)$ and a transverse splitting functional $\mathcal{F}_2[f''] = \sum m c_m f''(2m)$. The two-shell condition $f''(6) = \frac{1}{48}f''(4)$ leaves an unavoidable residual transverse gap $\mathcal{F}_2 = -f''(4) \neq 0$. In the three-shell model $(S_2 \cup S_3 \cup S_4)$, we find a unique strictly positive curvature ray $f''(4) : f''(6) : f''(8) = 1080 : 45 : 1$ that eliminates both orders simultaneously. Using the Ramanujan theta relation $q \frac{d}{dq} \Delta^2 = 2E_2 \Delta^2$, we prove that multi-Gaussian potentials achieve simultaneous infinite-crystal ultraisotropy through order $\|\mathbf{k}\|^{12}$. Finally, we formalize the Cauchy–Lamé reduction in Lean 4 and provide Python verification scripts for all dynamical spectra.

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1 Introduction

The Leech lattice $\Lambda_{24} \subset \mathbb{R}^{24}$ represents an extraordinary confluence of sphere packing optimality, exceptional group theory, and modular forms [2, 1]. Normalized such that its minimal non-zero squared norm is $\min_{\mathbf{R} \in \Lambda_{24} \setminus \{0\}} \|\mathbf{R}\|^2 = 4$, Λ_{24} is the unique even unimodular lattice in dimension 24 containing no roots. Its universal optimality, established by Cohn, Kumar, Miller, Radchenko, and Viazovska [1], proves that Λ_{24} minimizes potential energy among all unit-density 24-dimensional point configurations for every completely monotonic function of squared Euclidean distance.

In geometric analysis, every non-empty metric shell of Λ_{24} forms a spherical 11-design [9]. In lattice dynamics, this design property forces discrete lattice sums in the Born–von Kármán expansion to match their continuous $O(24)$ -spherical averages for all polynomial terms of degree at most 11, producing an acoustic tensor that is strictly isotropic through order $\|\mathbf{k}\|^8$.

The purpose of this paper is to establish the complete geometric, harmonic, and dynamical architecture of Λ_{24} from its contact vectors to infinity. We address five foundational questions:

1. **Polytopal Voronoi Facets:** While literature frequently cites the kissing number 196,560, we determine the complete facet census of $\text{Vor}(\Lambda_{24})$ and prove why vectors of norm $\|\mathbf{R}\|^2 \geq 8$ are strictly excluded.
2. **Modular Channel Classification:** We establish the double-cusp isomorphism $S_{12+d}^0 \cong M_{d-12}$ on $\text{SL}_2(\mathbb{Z})$ governing all harmonic shell moments, and we prove why degree-14 moments vanish on every shell.
3. **Completed L -Function Positivity:** We evaluate the Fourier coefficients of Δ^2 to order 25, identify where sign alternation breaks down, and prove via Mellin transformation that $L(\Delta^2, s) > 0$ for all $s > 0$.
4. **Acoustic Elasticity and Cauchy Reduction:** We formalize the macroscopic elasticity under virial prestress and prove that zero-stress equilibrium algebraically forces $\lambda = \mu = S_2$, fixing the sound velocity ratio to $v_L/v_T = \sqrt{3}$.
5. **Higher-Order Tensor Anisotropy:** We derive the complete irreducible tensor decomposition at orders $\|\mathbf{k}\|^{10}$ and $\|\mathbf{k}\|^{12}$. We prove that while a two-shell model cancels order- $\|\mathbf{k}\|^{10}$ via $f''(6) = \frac{1}{48}f''(4)$, it leaves a residual transverse splitting at order $\|\mathbf{k}\|^{12}$. We then construct exact three-shell (1080 : 45 : 1) and infinite-crystal multi-Gaussian potentials that achieve simultaneous ultraisotropy through order $\|\mathbf{k}\|^{12}$.

2 Leech Lattice Geometry and Normalization

Let $\Lambda_{24} \subset \mathbb{R}^{24}$ denote the standard even unimodular Leech lattice normalized such that:

$$\min_{\mathbf{R} \in \Lambda_{24} \setminus \{0\}} \|\mathbf{R}\|^2 = 4. \quad (1)$$

Because $\det(\Lambda_{24}) = 1$, the volume of the fundamental unit cell is $v_c = 1$. The equilibrium mass density with unit particle mass $M = 1$ is $\rho = M/v_c = 1$. The reciprocal lattice is $\Lambda_{\text{rec}} = 2\pi\Lambda_{24}^* = 2\pi\Lambda_{24}$, and the first Brillouin zone is the Voronoi cell $\mathcal{B}_{24} := \mathbb{R}^{24}/(2\pi\Lambda_{24})$.

The theta series of Λ_{24} is:

$$\Theta_{\Lambda_{24}}(q) = \sum_{m=0}^{\infty} |S_m| q^m = E_{12}(q) - \frac{65520}{691} \Delta(q) = 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + \dots, \quad (2)$$

where $S_m := \{\mathbf{R} \in \Lambda_{24} : \|\mathbf{R}\|^2 = 2m\}$. Because Λ_{24} contains no roots, $S_1 = \emptyset$ ($|S_1| = 0$). For every $m \geq 2$, the exact shell population is:

$$|S_m| = \frac{65520}{691} (\sigma_{11}(m) - \tau(m)), \quad (3)$$

where $\sigma_{11}(m) = \sum_{d|m} d^{11}$ and $\tau(m)$ is Ramanujan's tau function.

3 Exact Voronoi Facet Geometry of $\text{Vor}(\Lambda_{24})$

The Voronoi cell of Λ_{24} is the convex polytope:

$$\text{Vor}(\Lambda_{24}) := \left\{ \mathbf{x} \in \mathbb{R}^{24} : \mathbf{x} \cdot \mathbf{R} \leq \frac{1}{2} \|\mathbf{R}\|^2 \quad \forall \mathbf{R} \in \Lambda_{24} \setminus \{\mathbf{0}\} \right\}. \quad (4)$$

A vector $\mathbf{v} \in \Lambda_{24} \setminus \{\mathbf{0}\}$ is *Voronoi-relevant* if its bounding hyperplane

$$H_{\mathbf{v}} := \left\{ \mathbf{x} \in \mathbb{R}^{24} : \mathbf{x} \cdot \mathbf{v} = \frac{1}{2} \|\mathbf{v}\|^2 \right\}$$

intersects $\text{Vor}(\Lambda_{24})$ in a 23-dimensional facet

$$F_{\mathbf{v}} = \text{Vor}(\Lambda_{24}) \cap H_{\mathbf{v}}.$$

3.1 Covering-Radius Upper Bound

Theorem 3.1 (Relevance Norm Bound). *If $\mathbf{v} \in \Lambda_{24}$ is Voronoi-relevant, its squared Euclidean norm satisfies:*

$$\|\mathbf{v}\|^2 < 8. \quad (5)$$

Consequently, no vector of squared norm $\|\mathbf{v}\|^2 \geq 8$ can be Voronoi-relevant.

Proof. The orthogonal distance from the origin to $H_{\mathbf{v}}$ is

$$d(0, H_{\mathbf{v}}) = \frac{1}{2} \|\mathbf{v}\|,$$

with the projection located at $\mathbf{p} = \mathbf{v}/2$.

By Conway, Parker, and Sloane [3], the covering radius of the Leech lattice is $R_{\text{cov}} = \sqrt{2}$. Therefore,

$$\text{Vor}(\Lambda_{24}) \subset \overline{B}(0, \sqrt{2}).$$

- If

$$\frac{1}{2} \|\mathbf{v}\| > \sqrt{2} \iff \|\mathbf{v}\|^2 > 8,$$

then

$$H_{\mathbf{v}} \cap B(0, \sqrt{2}) = \emptyset,$$

so

$$H_{\mathbf{v}} \cap \text{Vor}(\Lambda_{24}) = \emptyset.$$

- If

$$\frac{1}{2} \|\mathbf{v}\| = \sqrt{2} \iff \|\mathbf{v}\|^2 = 8,$$

then $H_{\mathbf{v}}$ is tangent to the circumscribed sphere $S^{23}(\sqrt{2})$ at $\mathbf{v}/2$. Thus

$$H_{\mathbf{v}} \cap \overline{B}(0, \sqrt{2}) = \{\mathbf{v}/2\}.$$

Consequently,

$$H_{\mathbf{v}} \cap \text{Vor}(\Lambda_{24}) \subseteq \{\mathbf{v}/2\},$$

which has affine dimension 0. It cannot form a 23-dimensional facet.

Because Λ_{24} is an even lattice, $\|\mathbf{v}\|^2$ is an even integer. The only possible squared norms strictly less than 8 and at least the minimal norm 4 are $\|\mathbf{v}\|^2 = 4$ and $\|\mathbf{v}\|^2 = 6$. $\square$

3.2 Voronoi Coset Uniqueness and Facet Census

Theorem 3.2 (Voronoi (1908) [10]). *A vector $\mathbf{v} \in \Lambda \setminus \{\mathbf{0}\}$ is Voronoi-relevant if and only if $\pm\mathbf{v}$ are the strictly unique shortest vectors in the affine coset $\mathbf{v} + 2\Lambda$:*

$$\|\mathbf{v} - 2\mathbf{u}\|^2 \geq \|\mathbf{v}\|^2 \quad \forall \mathbf{u} \in \Lambda, \quad \text{with equality iff } \mathbf{u} \in \{\mathbf{0}, \mathbf{v}\}. \quad (6)$$

Theorem 3.3 (Total Facet Census of $\text{Vor}(\Lambda_{24})$). *The Voronoi polytope of the Leech lattice possesses **exactly** 16,969,680 **facets**, partitioned into two Co_0 -orbits:*

1. 196,560 **Type-I facets**: corresponding to minimal vectors $\mathbf{v} \in S_2$ ($\|\mathbf{v}\|^2 = 4$), situated at distance $d_1 = 1$ from the origin.
2. 16,773,120 **Type-II facets**: corresponding to second-shell vectors $\mathbf{v} \in S_3$ ($\|\mathbf{v}\|^2 = 6$), situated at distance $d_2 = \sqrt{3/2} \approx 1.2247$ from the origin.

Proof. Expanding the coset norm:

$$\|\mathbf{v} - 2\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + 4\left(\|\mathbf{u}\|^2 - \mathbf{v} \cdot \mathbf{u}\right). \quad (7)$$

We evaluate Voronoi's criterion for both candidate shells:

1. For $\mathbf{v} \in S_2$ ($\|\mathbf{v}\|^2 = 4$):

If $\mathbf{u} \notin \{\mathbf{0}, \mathbf{v}\}$, Cauchy–Schwarz gives

$$\mathbf{v} \cdot \mathbf{u} \leq 2\|\mathbf{u}\|.$$

Since the minimum nonzero squared norm is 4, if $\mathbf{u} \neq \mathbf{0}$ then $\|\mathbf{u}\| \geq 2$. Thus

$$\mathbf{v} \cdot \mathbf{u} \leq \|\mathbf{u}\|^2,$$

with equality only when $\mathbf{u} = \mathbf{v}$.

Because Λ_{24} is an even lattice,

$$\|\mathbf{u}\|^2 - \mathbf{v} \cdot \mathbf{u} \geq 1$$

for all $\mathbf{u} \notin \{\mathbf{0}, \mathbf{v}\}$. Equation (7) therefore yields

$$\|\mathbf{v} - 2\mathbf{u}\|^2 \geq 4 + 4(1) = 8 > 4.$$

Thus all 196,560 vectors of S_2 are Voronoi-relevant.

2. For $\mathbf{v} \in S_3$ ($\|\mathbf{v}\|^2 = 6$):

Suppose there exists $\mathbf{u} \notin \{\mathbf{0}, \mathbf{v}\}$ such that

$$\|\mathbf{u}\|^2 - \mathbf{v} \cdot \mathbf{u} \leq 0.$$

Then

$$\mathbf{v} \cdot \mathbf{u} \geq \|\mathbf{u}\|^2.$$

By Cauchy–Schwarz,

$$\|\mathbf{u}\|^2 \leq \mathbf{v} \cdot \mathbf{u} \leq \sqrt{6}\|\mathbf{u}\|,$$

hence

$$\|\mathbf{u}\|^2 \leq 6.$$

- If $\|\mathbf{u}\|^2 = 4$, then

$$\mathbf{v} \cdot \mathbf{u} \geq 4.$$

Hence

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v} \cdot \mathbf{u}) + \|\mathbf{u}\|^2 \leq 6 - 8 + 4 = 2.$$

But Λ_{24} has no vectors of squared norm 2.

- If $\|\mathbf{u}\|^2 = 6$, then

$$\mathbf{v} \cdot \mathbf{u} \geq 6.$$

Since

$$\|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 = 6,$$

equality in Cauchy–Schwarz forces $\mathbf{u} = \mathbf{v}$.

Hence, for every $\mathbf{u} \in \Lambda_{24} \setminus \{\mathbf{0}, \mathbf{v}\}$,

$$\|\mathbf{u}\|^2 - \mathbf{v} \cdot \mathbf{u} \geq 1,$$

and therefore

$$\|\mathbf{v} - 2\mathbf{u}\|^2 \geq 6 + 4(1) = 10 > 6.$$

Thus $\pm\mathbf{v}$ are strictly unique in $\mathbf{v} + 2\Lambda_{24}$, making all 16,773,120 vectors of S_3 Voronoi-relevant.

By Theorem 3.1, no other norms contribute. Therefore the total facet count is

$$|S_2| + |S_3| = 196,560 + 16,773,120 = 16,969,680.$$

□

Remark 3.4 (Voronoi Polytope versus Contact Polytope). It is essential to distinguish the facets of the Voronoi cell $\text{Vor}(\Lambda_{24})$ from the facets of the contact polytope (or kissing polytope) $\text{conv}(S_2)$.

The Voronoi cell $\text{Vor}(\Lambda_{24})$ is a space-filling parallelotope whose codimension-1 facets correspond bijectively to the Voronoi-relevant vectors (the 16,969,680 vectors of $S_2 \cup S_3$).

In contrast, the contact polytope

$$\mathcal{P}_{\text{contact}} := \text{conv}(S_2)$$

is the 24-dimensional convex hull of the 196,560 minimal vectors on S^{23} .

As demonstrated by Dutour Sikirić, Schürmann, and Vallentin [4], $\text{conv}(S_2)$ possesses an astronomical number of facets (exceeding 10^{13} facets partitioned into distinct Co_0 -orbits), which represent the spherical Delaunay caps and boundary faces of the packing. The exact census 16,969,680 in Theorem 3.3 applies strictly to the periodic Voronoi cell $\text{Vor}(\Lambda_{24})$.

4 Harmonic Multipoles to Infinity: $S_{12+d}^0 \cong M_{d-12}$

Let $\text{Harm}_d(\mathbb{R}^{24})$ denote the space of homogeneous harmonic polynomials of degree d . For $P_d \in \text{Harm}_d(\mathbb{R}^{24})$, the weighted theta series is

$$\Theta_{\Lambda_{24}, P_d}(\tau) := \sum_{\mathbf{R} \in \Lambda_{24}} P_d(\mathbf{R}) q^{\|\mathbf{R}\|^2/2}, \quad q = e^{2\pi i \tau}, \quad \tau \in \mathbb{H}. \quad (8)$$

By the Hecke–Schoeneberg theorem [5], for $d \geq 1$, $\Theta_{\Lambda_{24}, P_d}$ is a cusp form of weight $w = 12 + d$ on $\text{SL}_2(\mathbb{Z})$.

Because $\min_{\mathbf{R} \neq 0} \|\mathbf{R}\|^2 = 4$, the q^1 Fourier coefficient vanishes:

$$\Theta_{\Lambda_{24}, P_d}(\tau) = \sum_{m=2}^{\infty} \left(\sum_{\mathbf{R} \in S_m} P_d(\mathbf{R}) \right) q^m = a_2 q^2 + a_3 q^3 + \dots \implies a_1 = 0. \quad (9)$$

We define the double-cusp subspace:

$$S_w^0(\mathrm{SL}_2(\mathbb{Z})) := \{F \in S_w(\mathrm{SL}_2(\mathbb{Z})) : a_1(F) = 0\}. \quad (10)$$

Lemma 4.1 (Double-Cusp Factorization Isomorphism). *The mapping*

$$\boxed{\Phi : S_{12+d}^0(\mathrm{SL}_2(\mathbb{Z})) \xrightarrow{\sim} M_{d-12}(\mathrm{SL}_2(\mathbb{Z})), \quad F(\tau) \mapsto \frac{F(\tau)}{\Delta^2(\tau)}} \quad (11)$$

is an isomorphism of complex vector spaces.

Proof. The discriminant $\Delta(\tau) \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ has a simple zero at the cusp $\tau = i\infty$ and has no zeros in the upper half-plane $\mathbb{H}$.

If $F \in S_w^0$, then

$$F(\tau) = a_2 q^2 + \mathcal{O}(q^3).$$

The quotient

$$\frac{F(\tau)}{\Delta(\tau)} = a_2 q + \mathcal{O}(q^2)$$

is holomorphic on $\mathbb{H}$ and vanishes at ∞ , so $F/\Delta \in S_{w-12}(\mathrm{SL}_2(\mathbb{Z}))$.

Dividing by Δ a second time yields

$$\frac{F(\tau)}{\Delta(\tau)^2} = a_2 + \mathcal{O}(q),$$

which is holomorphic on $\mathbb{H}$ and at ∞ . Hence

$$F/\Delta^2 \in M_{w-24}(\mathrm{SL}_2(\mathbb{Z})) = M_{d-12}(\mathrm{SL}_2(\mathbb{Z})).$$

The inverse map $G \mapsto G\Delta^2$ is linear and injective. □

Theorem 4.2 (Rigorous Degree-14 Shell Vanishing). *For every shell $S_m \subset \Lambda_{24}$ ($m \geq 2$), and for every harmonic polynomial $P_{14} \in \mathrm{Harm}_{14}(\mathbb{R}^{24})$,*

$$\boxed{\sum_{\mathbf{R} \in S_m} P_{14}(\mathbf{R}) = 0.} \quad (12)$$

Proof. For $d = 14$, the modular weight is $w = 12 + 14 = 26$. By Lemma 4.1,

$$S_{26}^0(\mathrm{SL}_2(\mathbb{Z})) \cong M_{14-12}(\mathrm{SL}_2(\mathbb{Z})) = M_2(\mathrm{SL}_2(\mathbb{Z})).$$

Because there are no non-zero holomorphic modular forms of weight 2 on the full modular group,

$$M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\},$$

we have

$$S_{26}^0(\mathrm{SL}_2(\mathbb{Z})) = \{0\}.$$

While the full cusp space S_{26} is non-zero with $\dim S_{26} = 1$, its generator has $a_1 \neq 0$ and therefore does not belong to S_{26}^0 .

Since

$$\Theta_{\Lambda_{24}, P_{14}} \in S_{26}^0(\mathrm{SL}_2(\mathbb{Z})),$$

it is identically zero:

$$\Theta_{\Lambda_{24}, P_{14}}(\tau) = \sum_{m=2}^{\infty} \left(\sum_{\mathbf{R} \in S_m} P_{14}(\mathbf{R}) \right) q^m \equiv 0.$$

Therefore

$$\sum_{\mathbf{R} \in S_m} P_{14}(\mathbf{R}) = 0 \quad \forall m \geq 2.$$

□

5 Degree-12 Fourier Coefficients and Completed L -Function

For $d = 12$, $M_0(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}$, so $\dim S_{24}^0(\mathrm{SL}_2(\mathbb{Z})) = 1$.

The unique generator is

$$\Delta^2(\tau) = \sum_{m=2}^{\infty} c_m(\Delta^2) q^m,$$

where

$$c_m(\Delta^2) = \sum_{j=1}^{m-1} \tau(j) \tau(m-j).$$

Consequently, on every shell:

$$M_m(12) = c_m(\Delta^2) M_2(12). \quad (13)$$

m	$c_m(\Delta^2)$	Sign	m	$c_m(\Delta^2)$	Sign
2	+1	+	12	-424,520,544	-
3	-48	-	13	+1,268,350,272	+
4	+1,080	+	14	-1,211,937,160	-
5	-15,040	-	15	-4,306,546,080	- (Consecutive -)
6	+143,820	+	16	+18,293,091,840	+
7	-985,824	-	17	-23,522,231,424	-
8	+4,857,920	+	18	-26,299,018,683	- (Consecutive -)
9	-16,295,040	-	19	+137,218,594,320	+
10	+28,412,910	+	20	-150,999,182,320	-
11	+38,671,600	+	21	-134,713,340,160	- (Consecutive -)

Table 1: Exact Fourier coefficients $c_m(\Delta^2)$ showing the breakdown of sign alternation.

Theorem 5.1 (Strict Positivity of the Completed L -Function). *The completed L -function of Δ^2 :*

$$\Lambda(s) := (2\pi)^{-s} \Gamma(s) L(\Delta^2, s) = \int_1^{\infty} \Delta^2(iy) (y^{s-1} + y^{23-s}) dy \quad (14)$$

satisfies $\Lambda(s) > 0$ for all real $s \in \mathbb{R}$. Consequently,

$$L(\Delta^2, s) > 0 \quad \text{for all real } s > 0.$$

Proof. For $y > 0$,

$$\Delta(iy) = e^{-2\pi y} \prod_{n=1}^{\infty} (1 - e^{-2\pi n y})^{24} > 0,$$

so $\Delta^2(iy) > 0$.

Using the modular transformation

$$\Delta^2(i/y) = y^{24} \Delta^2(iy)$$

and splitting the Mellin integral at $y = 1$ yields the displayed representation.

For every $s \in \mathbb{R}$ and $y \geq 1$,

$$y^{s-1} + y^{23-s} > 0.$$

Because the integrand is strictly positive on $[1, \infty)$,

$$\Lambda(s) > 0.$$

Since $\Gamma(s) > 0$ for $s > 0$,

$$L(\Delta^2, s) = \frac{(2\pi)^s}{\Gamma(s)} \Lambda(s) > 0.$$

□

Corollary 5.2. *No single power-law pair potential $V(r) = Cr^{-2s}$ ($s > 0$) can eliminate the order- $\|\mathbf{k}\|^{10}$ degree-12 anisotropy of the infinite Leech crystal.*

6 Continuum Phonon Dynamics and Cauchy–Lamé Reduction

Consider particles interacting via an admissible pair potential $V(r) = f(r^2)$.

The Born–von Kármán dynamical matrix $D(\mathbf{k}) \in \text{Mat}_{24}(\mathbb{C})$ is:

$$D_{ab}(\mathbf{k}) := \sum_{\mathbf{R} \in \Lambda_{24} \setminus \{\mathbf{0}\}} (1 - \cos(\mathbf{k} \cdot \mathbf{R})) \left[2\delta_{ab} f'(\|\mathbf{R}\|^2) + 4R_a R_b f''(\|\mathbf{R}\|^2) \right]. \quad (15)$$

Expanding

$$1 - \cos(\mathbf{k} \cdot \mathbf{R}) = \frac{1}{2}(\mathbf{k} \cdot \mathbf{R})^2 + O(\|\mathbf{k}\|^4),$$

the quadratic acoustic tensor is:

$$D_{ab}^{(2)}(\mathbf{k}) = \sum_{c,d=1}^{24} C_{acbd} k_c k_d, \quad (16)$$

where

$$C_{acbd} = \sum_{\mathbf{R} \neq \mathbf{0}} R_c R_d \left[\delta_{ab} f'(\|\mathbf{R}\|^2) + 2R_a R_b f''(\|\mathbf{R}\|^2) \right]. \quad (17)$$

Define the radial shell sums:

$$S_1 := \frac{1}{24} \sum_{\mathbf{R} \neq \mathbf{0}} \|\mathbf{R}\|^2 f'(\|\mathbf{R}\|^2), \quad (18)$$

and

$$S_2 := \frac{1}{312} \sum_{\mathbf{R} \neq \mathbf{0}} \|\mathbf{R}\|^4 f''(\|\mathbf{R}\|^2). \quad (19)$$

Theorem 6.1 (Design-Forced Continuum Isotropy). *The spherical 11-design property forces the quadratic acoustic tensor to be strictly $O(24)$ -isotropic:*

$$D_{ab}^{(2)}(\mathbf{k}) = (S_1 + S_2) \|\mathbf{k}\|^2 \delta_{ab} + 2S_2 k_a k_b. \quad (20)$$

Theorem 6.2 (Prestress and Cauchy–Lamé Reduction). *Let f be an admissible central pair potential.*

1. **Virial Prestress:** *The hydrostatic Cauchy prestress is*

$$P_0 = -S_1 = -\frac{1}{24} \sum_{\mathbf{R} \neq \mathbf{0}} \|\mathbf{R}\|^2 f'(\|\mathbf{R}\|^2).$$

2. **Acoustic Moduli Under Prestress:** *Matching*

$$D_{ab}^{(2)}(\mathbf{k}) = \mu_{\text{eff}} \|\mathbf{k}\|^2 \delta_{ab} + (\lambda_{\text{eff}} + \mu_{\text{eff}}) k_a k_b$$

gives

$$\mu_{\text{eff}} = S_1 + S_2 = S_2 - P_0, \quad \lambda_{\text{eff}} = S_2 - S_1 = S_2 + P_0. \quad (21)$$

3. **Zero-Stress Cauchy Reduction:** *At mechanical zero-stress equilibrium ($P_0 = -S_1 = 0$), the Born stiffness tensor is completely symmetric under all index permutations. In particular, the Cauchy relation holds:*

$$\boxed{\lambda = \mu = S_2 = \frac{1}{312} \sum_{\mathbf{R} \in \Lambda_{24} \setminus \{\mathbf{0}\}} \|\mathbf{R}\|^4 f''(\|\mathbf{R}\|^2).} \quad (22)$$

The zero-stress acoustic response is governed by a single independent modulus μ .

Theorem 6.3 (Zero-Stress Sound Velocity Ratio). *At zero-stress equilibrium ($S_1 = 0$) with $S_2 > 0$ and density $\rho = 1$:*

$$v_T = \sqrt{\mu/\rho} = \sqrt{S_2}, \quad v_L = \sqrt{(\lambda + 2\mu)/\rho} = \sqrt{3S_2}. \quad (23)$$

Consequently,

$$\boxed{\frac{v_L}{v_T} = \sqrt{3} \approx 1.732050807568877 \dots} \quad (24)$$

7 Design-Controlled Ultraisotropy Through $\|\mathbf{k}\|^8$

Expanding

$$D(\mathbf{k}) = \sum_{j=1}^{\infty} D^{(2j)}(\mathbf{k}),$$

the order- $\|\mathbf{k}\|^{2j}$ coefficient is:

$$D_{ab}^{(2j)}(\mathbf{k}) = \frac{(-1)^{j-1}}{(2j)!} \sum_{\mathbf{R} \neq \mathbf{0}} (\mathbf{k} \cdot \mathbf{R})^{2j} \left[2\delta_{ab} f'(\|\mathbf{R}\|^2) + 4R_a R_b f''(\|\mathbf{R}\|^2) \right]. \quad (25)$$

The summands contain homogeneous polynomials in $\mathbf{R}$ of degrees $2j$ and $2j + 2$.

Theorem 7.1 (Ultraisotropy Through Order $\|\mathbf{k}\|^8$). *For each $j \in \{1, 2, 3, 4\}$, the maximum polynomial degree is*

$$2j + 2 \leq 10 \leq 11.$$

By Venkov's theorem, all harmonic multipoles of degree $1 \leq p \leq 10$ vanish shell-by-shell upon summation. Thus:

$$\boxed{D_{ab}^{(2j)}(\mathbf{k}) = A_j \|\mathbf{k}\|^{2j} \delta_{ab} + B_j \|\mathbf{k}\|^{2j-2} k_a k_b, \quad j \in \{1, 2, 3, 4\}.} \quad (26)$$

In particular, all 23 transverse polarizations remain degenerate through order $\|\mathbf{k}\|^8$:

$$\omega_{T,1}(\mathbf{k}) = \dots = \omega_{T,23}(\mathbf{k}) = v_T q + \alpha_3 q^3 + \alpha_5 q^5 + \alpha_7 q^7 + O(q^9), \quad q = \|\mathbf{k}\|, \quad (27)$$

with explicit algebraic coefficients α_j .

8 Order $\|\mathbf{k}\|^{10}$ and the Exact Two-Shell Cancellation

At order $\|\mathbf{k}\|^{10}$ ($j = 5$), the f' term has degree $10 \leq 11$ and is isotropic. The f'' term has degree 12.

Using

$$R_a R_b (\mathbf{k} \cdot \mathbf{R})^{10} = \frac{1}{132} \partial_a \partial_b (\mathbf{k} \cdot \mathbf{R})^{12}$$

and the series prefactor $\frac{4}{10!}$:

$$D_{\text{aniso},ab}^{(10)}(\mathbf{k}) = \frac{1}{33 \cdot 10!} \mathcal{A}_{12}[f''] \partial_a \partial_b H_{12}^{(2)}(\mathbf{k}), \quad (28)$$

where

$$\mathcal{A}_{12}[f''] := \sum_{m=2}^{\infty} c_m(\Delta^2) f''(2m). \quad (29)$$

8.1 Minimal-Shell Non-12-Design Certificate: Exact Combinatorial Derivation

In the integer realization

$$\Lambda_{24}^{\text{int}} = \sqrt{8} \Lambda_{24},$$

vectors in the minimal shell X_{int} have squared norm 32.

The 196,560 vectors partition into three classical shapes:

- **Type 1:** $(\pm 4^2, 0^{22})$, multiplicity

$$2^2 \binom{24}{2} = 1,104.$$

- **Type 2:** $(\pm 2^8, 0^{16})$, supported on the 759 octads of the extended binary Golay code $\mathcal{G}_{24}$, multiplicity

$$2^7 \times 759 = 97,152.$$

- **Type 3:** $(\mp 3^1, \pm 1^{23})$, odd vectors with sign parity, multiplicity

$$24 \times 2^{12} = 98,304.$$

We evaluate the moment

$$M_{12}^{\text{int}}(\mathbf{u}) := \sum_{\mathbf{R} \in X_{\text{int}}} (\mathbf{u} \cdot \mathbf{R})^{12}$$

along

$$u_A = e_1$$

and

$$u_B = \frac{e_1 + e_2}{\sqrt{2}}.$$

1. Evaluation along $u_A = e_1$:

- *Type 1:* Coordinate 1 is non-zero in $2 \times 23 \times 2 = 92$ vectors, contributing

$$92 \times 4^{12} = 1,543,503,872.$$

- *Type 2:* In the Steiner system $S(5, 8, 24)$, the number of octads containing a fixed point is

$$759 \times \frac{8}{24} = 253.$$

With 128 sign configurations per octad,

$$253 \times 128 = 32,384$$

vectors have $R_1 = \pm 2$, contributing

$$32,384 \times 2^{12} = 132,644,864.$$

- *Type 3:* In 4,096 vectors, coordinate 1 is ± 3 :

$$4,096 \times 3^{12} = 2,176,782,336.$$

In the remaining

$$23 \times 4,096 = 94,208$$

vectors, coordinate 1 is ± 1 , contributing

$$94,208.$$

Thus Type 3 contributes

$$2,176,876,544.$$

Summing all three types:

$$M_{12}^{\text{int}}(u_A) = 1,543,503,872 + 132,644,864 + 2,176,876,544 = 3,853,025,280. \quad (30)$$

2. **Evaluation along $u_B = (e_1 + e_2)/\sqrt{2}$:**

Here

$$(\mathbf{u} \cdot \mathbf{R})^{12} = \frac{1}{64}(R_1 + R_2)^{12}.$$

- *Type 1:* Vectors with support on $\{1, 2\}$ contribute

$$2 \times 8^{12}.$$

Vectors with support on $\{1, k\}$ or $\{2, k\}$, $k \geq 3$, contribute

$$176 \times 4^{12}.$$

Dividing by 64 gives

$$2,193,620,992.$$

- *Type 2:* In $S(5, 8, 24)$, exactly

$$759 \times \frac{28}{276} = 77$$

octads contain both 1 and 2, and

$$2(253 - 77) = 352$$

octads contain exactly one of them. Summing over sign parities gives

$$1,294,729,216.$$

- *Type 3*: Summing over coordinate parities gives

$$1,076,887,552.$$

Hence

$$M_{12}^{\text{int}}(u_B) = 2,193,620,992 + 1,294,729,216 + 1,076,887,552 = 4,565,237,760. \quad (31)$$

Taking the difference yields the exact harmonic moment:

$$\boxed{H_{12}^{\text{int}}(u_B) - H_{12}^{\text{int}}(u_A) = M_{12}^{\text{int}}(u_B) - M_{12}^{\text{int}}(u_A) = 712,212,480 \neq 0.} \quad (32)$$

For

$$\mathbf{u} = \frac{1}{\sqrt{5}}(1, 2, 0, \dots),$$

the transverse Hessian splits into eigenspaces of multiplicities 1 and 22 with exact eigenvalues

$$\lambda_{\text{in}} = \frac{254700453888}{625}, \quad \lambda_{\text{out}} = \frac{209217945600}{625}, \quad (33)$$

giving the exact non-zero gap:

$$\Delta\lambda_{\text{int}} = \lambda_{\text{in}} - \lambda_{\text{out}} = \frac{45482508288}{625} \neq 0. \quad (34)$$

8.2 Exact Two-Shell Cancellation

Restricting Eq. (28) to the first two shells $S_2 \cup S_3$ ($c_2 = 1$, $c_3 = -48$):

$$\mathcal{A}_{12}^{(2\text{-shell})} = f''(4) - 48f''(6) = 0. \quad (35)$$

Therefore

$$\boxed{f''(6) = \frac{1}{48}f''(4).} \quad (36)$$

Under this condition, the order- $\|\mathbf{k}\|^{10}$ anisotropic tensor vanishes identically.

9 Order $\|\mathbf{k}\|^{12}$ Irreducible Tensor Decomposition

At $j = 6$,

$$D_{ab}^{(12)}(\mathbf{k}) = T_{ab}^{(12,1)}(\mathbf{k}) + T_{ab}^{(12,2)}(\mathbf{k}).$$

Theorem 9.1 (Harmonic Decomposition at Order $\|\mathbf{k}\|^{12}$). *The order- $\|\mathbf{k}\|^{12}$ anisotropic dynamical matrix is:*

$$\boxed{D_{\text{aniso},ab}^{(12)}(\mathbf{k}) = -\frac{1}{12!} \left\{ 2 \left(\mathcal{F}_1[f'] + \frac{1}{12} \mathcal{F}_2[f''] \right) \delta_{ab} H_{12}^{(2)}(\mathbf{k}) \right.} \\ \left. + \frac{1}{6} \mathcal{F}_2[f''] \left(k_a \partial_b H_{12}^{(2)}(\mathbf{k}) + k_b \partial_a H_{12}^{(2)}(\mathbf{k}) \right) \right.} \\ \left. + \frac{1}{12} \mathcal{F}_2[f''] \|\mathbf{k}\|^2 \partial_a \partial_b H_{12}^{(2)}(\mathbf{k}) \right\}. \quad (37)$$

The two independent modular functionals are

$$\boxed{\mathcal{F}_1[f'] = \sum_{m=2}^{\infty} c_m(\Delta^2) f'(2m), \quad \mathcal{F}_2[f''] = \sum_{m=2}^{\infty} c_m(\Delta^2) m f''(2m).} \quad (38)$$

Proof. For the f' term, the prefactor is

$$-\frac{2}{12!}\delta_{ab}.$$

The degree-12 harmonic projection of $(\mathbf{k} \cdot \mathbf{R})^{12}$ gives

$$-\frac{2}{12!}\delta_{ab}H_{12}^{(m)}(\mathbf{k}).$$

Summing over shells yields

$$-\frac{2}{12!}\mathcal{F}_1[f']\delta_{ab}H_{12}^{(2)}(\mathbf{k}).$$

For the f'' term,

$$R_a R_b (\mathbf{k} \cdot \mathbf{R})^{12} = \frac{1}{182} \partial_a \partial_b (\mathbf{k} \cdot \mathbf{R})^{14}.$$

We expand

$$(\mathbf{k} \cdot \mathbf{R})^{14} = H_{14}(\mathbf{k}, \mathbf{R}) + \|\mathbf{R}\|^2 Q_{12}(\mathbf{k}, \mathbf{R}) + \dots$$

Applying the Laplacian $\Delta_{\mathbf{R}}$:

$$\Delta_{\mathbf{R}}(\mathbf{k} \cdot \mathbf{R})^{14} = 182 \|\mathbf{k}\|^2 (\mathbf{k} \cdot \mathbf{R})^{12}, \quad (39)$$

$$\Delta_{\mathbf{R}} \left[\|\mathbf{R}\|^2 Q_{12} \right] = 2(2 \times 12 + 24) Q_{12} = 96 Q_{12}. \quad (40)$$

Thus

$$Q_{12} = \frac{182}{96} \|\mathbf{k}\|^2 H_{12} = \frac{91}{48} \|\mathbf{k}\|^2 H_{12}.$$

On shell S_m , $\|\mathbf{R}\|^2 = 2m$, and $H_{14}^{(m)} \equiv 0$ by Theorem 4.2. Hence:

$$\left[\sum_{\mathbf{R} \in S_m} R_a R_b (\mathbf{k} \cdot \mathbf{R})^{12} \right]_{\text{aniso}} = \frac{1}{182} \partial_a \partial_b \left[(2m) \frac{91}{48} \|\mathbf{k}\|^2 H_{12}^{(m)} \right] = \frac{m}{48} \partial_a \partial_b \left[\|\mathbf{k}\|^2 H_{12}^{(m)} \right]. \quad (41)$$

Multiplying by $-\frac{4}{12!}$ yields

$$-\frac{1}{12 \cdot 12!} m \partial_a \partial_b [\|\mathbf{k}\|^2 H_{12}^{(m)}].$$

Expanding

$$\partial_a \partial_b [\|\mathbf{k}\|^2 H_{12}] = 2\delta_{ab} H_{12} + 2(k_a \partial_b H_{12} + k_b \partial_a H_{12}) + \|\mathbf{k}\|^2 \partial_a \partial_b H_{12},$$

and combining with the f' term gives Eq. (37). $\square$

9.1 Generic Transverse Splitting

For any unit transverse polarization $\mathbf{v} \in \mathbf{k}^\perp$,

$$\mathbf{v} \cdot \mathbf{k} = 0, \quad \|\mathbf{v}\| = 1,$$

we have

$$\mathbf{v}^\top D_{\text{aniso}}^{(12)}(\mathbf{k}) \mathbf{v} = -\frac{1}{12!} \left[2\mathcal{F}_1 + \frac{1}{6}\mathcal{F}_2 \right] H_{12}^{(2)}(\mathbf{k}) - \frac{\|\mathbf{k}\|^2}{12 \cdot 12!} \mathcal{F}_2[f''] \mathbf{v}^\top (\nabla \nabla H_{12}^{(2)}(\mathbf{k})) \mathbf{v}. \quad (42)$$

Corollary 9.2. *The order- $\|\mathbf{k}\|^{12}$ transverse polarization splitting is governed strictly by $\mathcal{F}_2[f'']$:*

$$\Delta \lambda_T^{(12)}(\mathbf{k}) = \frac{\|\mathbf{k}\|^2}{12 \cdot 12!} \mathcal{F}_2[f''] \left[\mathbf{v}_2^\top (\nabla \nabla H_{12}^{(2)}) \mathbf{v}_2 - \mathbf{v}_1^\top (\nabla \nabla H_{12}^{(2)}) \mathbf{v}_1 \right]. \quad (43)$$

Generic transverse acoustic branch splitting occurs at order $\|\mathbf{k}\|^{12}$ whenever $\mathcal{F}_2[f''] \neq 0$.

10 Multiscale Annihilation: Two-Shell, Three-Shell, and Multi-Gaussian

10.1 Residual Splitting in the Two-Shell Model

In the two-shell model ($S_2 \cup S_3$):

$$\mathcal{A}_{12} = f''(4) - 48f''(6) = 0 \implies f''(6) = \frac{1}{48}f''(4). \quad (44)$$

Evaluating $\mathcal{F}_2$ under this condition:

$$\mathcal{F}_2 = 2f''(4) - 144f''(6) \quad (45)$$

$$= 2f''(4) - 144 \left(\frac{1}{48}f''(4) \right) \quad (46)$$

$$= 2f''(4) - 3f''(4) \quad (47)$$

$$= \boxed{-f''(4) \neq 0}. \quad (48)$$

Cancelling order- $\|\mathbf{k}\|^{10}$ leaves a transverse splitting at order $\|\mathbf{k}\|^{12}$.

10.2 Three-Shell Simultaneous Cancellation Ray

Including shell S_4 ($c_4 = 1080$):

$$\begin{cases} f''(4) - 48f''(6) + 1080f''(8) = 0, \\ 2f''(4) - 144f''(6) + 4320f''(8) = 0. \end{cases} \quad (49)$$

Subtracting $2 \times$ (Eq. 1) from Eq. 2 gives

$$-48f''(6) + 2160f''(8) = 0,$$

hence

$$f''(6) = 45f''(8).$$

Substituting back gives

$$f''(4) = 1080f''(8).$$

Thus:

$$\boxed{f''(4) : f''(6) : f''(8) = 1080 : 45 : 1 \iff \left(1 : \frac{1}{24} : \frac{1}{1080} \right)}. \quad (50)$$

All three curvature coefficients are positive.

10.3 Infinite-Crystal Multi-Gaussian Ultraisotropy

For a multi-Gaussian interaction

$$V(r) = \sum_{i=1}^N A_i e^{-\alpha_i r^2},$$

with

$$q_i = e^{-2\alpha_i},$$

the Ramanujan theta relation

$$q \frac{d}{dq} \Delta^2 = 2E_2 \Delta^2$$

evaluates the infinite-crystal functionals:

$$\mathcal{A}_{12}[f''] = \sum_{i=1}^N A_i \alpha_i^2 \Delta^2(q_i), \quad (51)$$

$$\mathcal{F}_1[f'] = - \sum_{i=1}^N A_i \alpha_i \Delta^2(q_i), \quad (52)$$

$$\mathcal{F}_2[f''] = 2 \sum_{i=1}^N A_i \alpha_i^2 E_2(q_i) \Delta^2(q_i). \quad (53)$$

Theorem 10.1. *For $N = 4$ with distinct parameters*

$$\boldsymbol{\alpha} = (0.5, 0.7, 1.0, 1.4),$$

the 3×4 coefficient matrix

$$M = \begin{pmatrix} \alpha_1^2 \Delta^2(q_1) & \cdots & \alpha_4^2 \Delta^2(q_4) \\ \alpha_1 \Delta^2(q_1) & \cdots & \alpha_4 \Delta^2(q_4) \\ 2\alpha_1^2 E_2(q_1) \Delta^2(q_1) & \cdots & 2\alpha_4^2 E_2(q_4) \Delta^2(q_4) \end{pmatrix} \quad (54)$$

evaluates numerically to:

$$M \approx \begin{pmatrix} 1.835435 \times 10^{-16} & 7.017995 \times 10^{-10} & 6.114176 \times 10^{-6} & 2.952235 \times 10^{-4} \\ 3.670870 \times 10^{-16} & 1.002571 \times 10^{-9} & 6.114176 \times 10^{-6} & 2.108739 \times 10^{-4} \\ -1.008697 \times 10^{-14} & -1.624051 \times 10^{-8} & -4.731888 \times 10^{-5} & -4.426624 \times 10^{-4} \end{pmatrix}. \quad (55)$$

Its singular values are strictly positive:

$$\sigma_1 \approx 5.738833 \times 10^{-4}, \quad \sigma_2 \approx 2.337850 \times 10^{-5}, \quad \sigma_3 \approx 1.630044 \times 10^{-10}. \quad (56)$$

The four 3×3 minors are non-zero, including

$$\det(M_{[:,\{0,1,2\}]}) \approx 3.807 \times 10^{-30}, \quad \det(M_{[:,\{1,2,3\}]}) \approx 2.187 \times 10^{-18}. \quad (57)$$

Consequently, $\text{rank } M = 3$, producing a unique 1-dimensional ray of multi-Gaussian amplitudes $\mathbf{A} \in \ker(M)$ under which the infinite Leech crystal is strictly isotropic through order $\|\mathbf{k}\|^{12}$:

$$\boxed{D_{\text{aniso}}^{(10)}(\mathbf{k}) \equiv 0 \quad \text{and} \quad D_{\text{aniso}}^{(12)}(\mathbf{k}) \equiv 0.} \quad (58)$$

11 Master Multipole Hierarchy Across Orders

The harmonic multipole decomposition of $D(\mathbf{k})$ across all long-wavelength orders is governed by the modular channel isomorphism $S_{12+d}^0 \cong M_{d-12}$:

12 Formal and Numerical Verification Artifacts

12.1 Lean 4 Formalization: Cauchy–Lamé Reduction

Listing 1 contains the machine-checked Lean 4 formalization (`LeechElasticity.lean`) proving that microscopic central-force Cauchy symmetry on an isotropic fourth-rank tensor forces $\lambda = \mu$, which in turn algebraically fixes $v_L^2/v_T^2 = 3$ over $\mathbb{Q}$.

Order	Harmonic Sectors	Modular Channel	Anisotropy Condition / Mechanism
$\ \mathbf{k}\ ^2 \dots \ \mathbf{k}\ ^8$	None (≤ 11)	$\{0\}$	Universally isotropic (Venkov 11-design)
$\ \mathbf{k}\ ^{10}$	$d = 12$	Δ^2	$\mathcal{A}_{12}[f''] = \sum_m c_m f''(2m) = 0$
$\ \mathbf{k}\ ^{12}$	$d = 12$ ($d = 14$ absent)	$\Delta^2, E_2\Delta^2$	$\mathcal{F}_1[f'] = 0$ and $\mathcal{F}_2[f''] = \sum_m m c_m f''(2m) = 0$ ($S_{26}^0 = \Delta^2 M_2 = \{0\}$ eliminates degree 14)
$\ \mathbf{k}\ ^{14}$	$d = 16$	$E_4\Delta^2$	$\mathcal{A}_{16}[f''] = \sum_m [q^m](E_4\Delta^2) f''(2m) = 0$
$\ \mathbf{k}\ ^{16}$	$d = 16, 18$	$E_4\Delta^2, E_6\Delta^2$	$S_{30}^0 \cong M_6 = \mathbb{C}E_6$
$\ \mathbf{k}\ ^{18}$	$d = 16, 18, 20$	$E_4^2\Delta^2$	$S_{32}^0 \cong M_8 = \mathbb{C}E_4^2$
$\ \mathbf{k}\ ^{20}$	$d = 16, \dots, 22$	$E_4E_6\Delta^2$	$S_{34}^0 \cong M_{10} = \mathbb{C}E_4E_6$
$\ \mathbf{k}\ ^{22}$	$d = 16, \dots, 24$	$E_4^3\Delta^2, \Delta^3$	$S_{36}^0 \cong M_{12} = \mathbb{C}E_4^3 \oplus \mathbb{C}\Delta$ (dim = 2)

Table 2: The master harmonic multipole hierarchy across all long-wavelength orders.

```

import Mathlib.Data.Rat.Basic
import Mathlib.Data.Real.Basic
import Mathlib.Tactic.Linarith
import Mathlib.Tactic.Ring

namespace LeechElasticity

/-- General isotropic elasticity tensor parameters (Lame coefficients) -/
structure IsotropicElasticity (alpha : Type*) where
  lambda_param : alpha
  mu_param : alpha

/-- Component evaluation of the isotropic 4th-rank stiffness tensor -/
def C_tensor (params : IsotropicElasticity ℚ) (i j k l : Fin 24) : ℚ :=
  let delta (a b : Fin 24) : ℚ := if a = b then 1 else 0
  params.lambda_param * (delta i j * delta k l) +
  params.mu_param * (delta i k * delta j l + delta i l * delta j k)

/-- The central-force Cauchy symmetry condition:  $C_{\{ikjl\}} = C_{\{ijkl\}}$  -/
def SatisfiesCauchy (params : IsotropicElasticity ℚ) : Prop :=
  ∀ i j k l : Fin 24, C_tensor params i k j l = C_tensor params i j k l

/-- THEOREM: For any isotropic medium, Cauchy symmetry forces  $\lambda = \mu$  -/
theorem cauchy_lame_reduction (params : IsotropicElasticity ℚ)
  (h : SatisfiesCauchy params) : params.lambda_param = params.mu_param := by
  have h_eval := h 0 1 0 1
  unfold C_tensor at h_eval
  have h00 : (if (0 : Fin 24) = 0 then (1 : ℚ) else 0) = 1 := rfl
  have h01 : (if (0 : Fin 24) = 1 then (1 : ℚ) else 0) = 0 := rfl
  have h11 : (if (1 : Fin 24) = 1 then (1 : ℚ) else 0) = 1 := rfl
  have h10 : (if (1 : Fin 24) = 0 then (1 : ℚ) else 0) = 0 := rfl
  revert h_eval
  simp only [h00, h01, h11, h10]
  intro h_eq
  linarith

/-- COROLLARY: The sound velocity squared ratio  $v_L^2 / v_T^2$  is strictly 3 -/
theorem sound_velocity_squared_ratio (params : IsotropicElasticity ℚ)
  (h_cauchy : SatisfiesCauchy params) (h_pos : params.mu_param > 0) :
  let v_L_sq := params.lambda_param + 2 * params.mu_param
  let v_T_sq := params.mu_param
  v_L_sq / v_T_sq = 3 := by
  have h_eq : params.lambda_param = params.mu_param :=
    cauchy_lame_reduction params h_cauchy
  dsimp
  rw [h_eq]
  have h_denom : params.mu_param ≠ 0 := by linarith
  calc

```

```

(params.mu_param + 2 * params.mu_param) / params.mu_param
= (3 * params.mu_param) / params.mu_param := by ring_nf
_ = 3 := mul_div_cancel_right0 3 h_denom

end LeechElasticity

```

Listing 1: Lean 4 formalization: `LeechElasticity.lean`.

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