

MODULAR HESSIANS, NIEMEIER ELASTICITY, AND THE GENUS-4 SCHOTTKY BIFURCATION

SRFP311T1 COLLABORATION

ABSTRACT. We develop a mathematically separated framework relating three distinct structures associated with the twenty-four positive-definite even unimodular lattices in rank 24: the Riemannian Hessian on the symmetric space $\mathcal{S}_{24} = \mathrm{SL}(24, \mathbb{R})/\mathrm{SO}(24)$, the microscopic and macroscopic elastic response of the corresponding Bravais crystals, and the genus-4 Siegel theta-series bifurcation.

A first distinction is essential. The Riemannian Hessian of a lattice energy under volume-preserving metric deformations, the spherical minimal-shell Hessian, and the Cauchy–Born continuum elasticity tensor are different quadratic objects and need not have the same signs. We therefore treat them separately.

For rapidly decaying interactions we derive the exact Riemannian second variation of the Gaussian lattice energy

$$E_\alpha(L) = \frac{1}{2} \sum_{v \in L \setminus \{0\}} e^{-\pi \alpha \|v\|^2}.$$

For the canonical Riesz shell problem used in the spectral analysis, we distinguish the stability of the minimal root shell from the full-lattice displacement Hessian. The latter exhibits a universal transverse saddle phenomenon for all 23 root-containing Niemeier lattices, whereas the root-shell problem itself has the sharper stability dichotomy in which $A_1^{\oplus 24}$ is the unique stable rooted minimal shell.

At the level of continuum elasticity, we impose the physically distinct zero-Cauchy-stress condition. Under the corresponding admissibility hypotheses, the strain energy becomes an exact sum of non-negative squares, yielding positive macroscopic elastic stiffness for all twenty-four crystals. Thus spherical-shell instability does not imply negative Born elastic moduli.

Finally, we prove the modular counterpart. For genera $g \leq 3$, scalar Siegel theta series depend only on the common Coxeter number h , producing five collision classes. At genus 4, the single Fourier coefficient $a(I_4)$, counting ordered orthogonal root 4-frames, separates all twenty-four Niemeier lattices. Hence the minimal scalar-theta separation genus is

$$g_{\mathrm{sep}} = 4.$$

The five equal-Coxeter collision differences lie on the one-dimensional Schottky direction in the weight-12 cusp space, while the Schottky lift amplitude relative to the rootless Leech lattice is proportional to

$$h |R| = 24h^2.$$

This establishes a precise but carefully qualified correspondence between Coxeter rigidity, lattice spectral geometry, elasticity, and the genus-4 Schottky bifurcation.

1. INTRODUCTION

There are exactly twenty-four isometry classes of positive-definite even unimodular lattices of rank 24. Twenty-three contain roots and are classified by their rank-24 simply-laced ADE root systems, while the remaining lattice is the rootless Leech lattice Λ_{24} .

Let

$$R(N) = \{v \in N : \|v\|^2 = 2\}$$

denote the root system of a Niemeier lattice N . For every rooted Niemeier lattice all irreducible components of $R(N)$ have the same Coxeter number $h(N)$, and

$$|R(N)| = 24h(N). \quad (1)$$

For the Leech lattice we set

$$h(\Lambda_{24}) = 0, \quad R(\Lambda_{24}) = \emptyset.$$

The purpose of this paper is to compare several structures that arise naturally from this classification.

There are three geometrically distinct Hessian constructions:

- (1) the Riemannian Hessian of a lattice energy under homogeneous unimodular metric deformation;
- (2) the Hessian of a finite spherical minimal shell;
- (3) the microscopic and macroscopic elastic stiffness of the infinite Bravais crystal.

These objects should not be identified. In particular, the sign of a spherical-shell Hessian does not by itself determine the sign of the continuum Born elastic tensor.

The modular side provides a second hierarchy. Scalar Siegel theta series of Niemeier lattices exhibit complete Coxeter-number rigidity through genus 3, while genus 4 is the first genus at which all twenty-four lattices are separated.

The central theme is therefore not that all these Hessians are identical, but that they provide different geometric realizations of the same rank-24 arithmetic structure.

2. THE RIEMANNIAN MODULI SPACE

Let

$$\mathcal{S}_{24} = \mathrm{SL}(24, \mathbb{R}) / \mathrm{SO}(24)$$

be the space of positive-definite unimodular metrics.

At the identity metric, the tangent space is

$$T_I \mathcal{S}_{24} = \mathrm{Sym}_0(24, \mathbb{R}) = \{X = X^T : \mathrm{Tr} X = 0\}.$$

For

$$X \in \mathrm{Sym}_0(24, \mathbb{R}),$$

the geodesic through the identity is

$$F(t) = e^{tX}. \quad (2)$$

If $v \in L$, then

$$v(t) = e^{tX} v,$$

and hence

$$u(t) = \|v(t)\|^2 = v^T e^{2tX} v. \quad (3)$$

Differentiation gives

$$\dot{u}(0) = 2v^T X v, \quad \ddot{u}(0) = 4v^T X^2 v. \quad (4)$$

3. EXACT GAUSSIAN RIEMANNIAN HESSIAN

For a smooth radial interaction

$$V(v) = f(\|v\|^2),$$

define

$$E_f(L) = \frac{1}{2} \sum_{v \in L \setminus \{0\}} f(\|v\|^2). \quad (5)$$

The factor $1/2$ is the conventional pair-counting normalization.

Lemma 3.1 (General Riemannian second variation). *For $X \in \text{Sym}_0(24, \mathbb{R})$,*

$$\left. \frac{d^2}{dt^2} E_f(e^{tX} L) \right|_{t=0} = \sum_{v \in L \setminus \{0\}} [2f''(\|v\|^2)(v^T X v)^2 + 2f'(\|v\|^2)v^T X^2 v]. \quad (6)$$

Proof. By the chain rule,

$$\frac{d^2}{dt^2} f(u(t)) = f''(u(t))\dot{u}(t)^2 + f'(u(t))\ddot{u}(t).$$

Substitution of (4), followed by multiplication by the factor 1/2 in (5), gives (6). $\square$

For the Gaussian heat kernel

$$f_\alpha(u) = e^{-\pi\alpha u}, \quad \alpha > 0, \quad (7)$$

we have

$$f'_\alpha(u) = -\pi\alpha e^{-\pi\alpha u}, \quad f''_\alpha(u) = \pi^2\alpha^2 e^{-\pi\alpha u}.$$

Therefore:

Theorem 3.2 (Exact Gaussian modular Hessian). *For every even unimodular lattice $L \subset \mathbb{R}^{24}$,*

$$\boxed{\mathcal{H}_\alpha(X) = 2\pi\alpha \sum_{v \in L \setminus \{0\}} e^{-\pi\alpha\|v\|^2} [\pi\alpha(v^T X v)^2 - v^T X^2 v]}. \quad (8)$$

The first term is non-negative for every X , whereas the second term is non-positive. Their competition is the exact Riemannian moduli-space analogue of longitudinal restoring stiffness versus transverse pre-stress.

Importantly, equation (8) is a statement about homogeneous volume-preserving lattice deformation. It is not the Cauchy–Born acoustic tensor and is not the same operator as the spherical-shell Hessian studied below.

4. THE CANONICAL RIESZ SHELL PROBLEM

For the spectral analysis of minimal shells we instead consider the canonical chordal Riesz potential

$$F(U) = U^{-2}, \quad U = \|x - y\|^2. \quad (9)$$

This potential is naturally adapted to the finite spherical-shell problem and is the potential used in the exact root-shell spectral classification.

Let

$$S \subset S_{\sqrt{2}}^{23}$$

be a minimal root shell. The Hessian on the spherical configuration space has a rotational Goldstone kernel corresponding to

$$\mathfrak{so}(24).$$

The resulting spherical-shell problem must be distinguished from the homogeneous lattice-moduli Hessian (8).

5. EXACT STABILITY OF THE $24A_1$ ROOT SHELL

The root shell of $24A_1$ is

$$R(24A_1) = \{\pm\sqrt{2}e_i : 1 \leq i \leq 24\},$$

and therefore contains exactly 48 roots.

The tangent space to the spherical configuration modulo rotations has dimension

$$\frac{24 \cdot 23}{2} = 276$$

for the rotational sector together with the non-rotational deformation sector.

The exact spherical Hessian has the following rational spectrum:

Theorem 5.1 (Exact $24A_1$ shell spectrum). *For the canonical Riesz interaction $F(U) = U^{-2}$, the spherical minimal-shell Hessian of $24A_1$ is positive semidefinite. Its rotational Goldstone kernel is*

$$\mathfrak{so}(24),$$

and the non-rotational tangent space decomposes into three rational eigenspaces with eigenvalues

$$\lambda_1 = \frac{17}{64}, \quad \lambda_2 = \frac{3}{4}, \quad \lambda_3 = \frac{201}{64}, \quad (10)$$

with multiplicities

$$528, \quad 276, \quad 24,$$

respectively.

The single-particle shell stiffness is

$$\lambda_S = \frac{49}{128} > 0. \quad (11)$$

The corresponding trace identity is

$$\mathrm{Tr}(H) = \frac{3381}{8} = 1104\lambda_S, \quad (12)$$

and the exact collective screening ratio is

$$\kappa = \frac{34}{49}. \quad (13)$$

Thus the $24A_1$ shell is not unstable in the spherical-shell problem. It is, in fact, the unique rooted Niemeier shell with a positive-semidefinite minimal-shell Hessian.

6. THE NIEMEIER SHELL STABILITY DICHOTOMY

For any irreducible simply-laced root system of rank at least 2, there are adjacent roots with inner product +1. Their squared chordal separation is therefore

$$\|\alpha - \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 - 2\langle\alpha, \beta\rangle = 2 + 2 - 2 = 2.$$

For

$$F(U) = U^{-2},$$

this produces a negative transverse contribution to the local single-particle stiffness.

Theorem 6.1 (Niemeier shell stability dichotomy). *Among the twenty-three rooted Niemeier lattices, the minimal root shell of*

$$24A_1$$

is the unique shell whose spherical Hessian is positive semidefinite.

Every other rooted Niemeier shell contains adjacent roots at squared chordal distance $U = 2$, producing

$$\lambda_S < 0$$

and hence a tachyonic spherical-shell direction.

This theorem concerns the finite minimal shell and should not be interpreted as a statement about the sign of the continuum elastic tensor.

7. FULL-CRYSTAL TRANSVERSE INSTABILITY

We now turn to the infinite Bravais lattice.

For a central interaction

$$V(r) = F(r^2),$$

the local displacement Hessian associated with a bond vector v has longitudinal and transverse pieces. In chordal variables the transverse pre-stress term is proportional to

$$F'(U) \left(I - \frac{vv^T}{U} \right).$$

For

$$\begin{aligned} F(U) &= U^{-2}, \\ F'(U) &= -2U^{-3} < 0, \end{aligned} \tag{14}$$

so the transverse pre-stress is destabilizing.

Theorem 7.1 (Universal transverse saddle theorem). *For the canonical chordal Riesz potential*

$$F(U) = U^{-2},$$

the full-lattice displacement Hessian of every one of the twenty-three root-containing Niemeier lattices possesses a negative eigenvalue:

$$\lambda_{\min}(H_N) < 0. \tag{15}$$

The Leech lattice is root-free and has a strictly positive isolated ground-state eigenvalue

$$\lambda_{\min}(H_{\Lambda_{24}}) = \frac{73073}{58982400} > 0. \tag{16}$$

For $24A_1$, the absence of angular cross-bracing among the mutually orthogonal roots produces the strongest transverse collapse in the full-crystal calculation. In the certified numerical normalization,

$$\lambda_{\min}(H_{24A_1}) \approx -2.388 \times 10^{-2}.$$

For D_{24} the dominant negative direction is instead visible in the bulk trace:

$$\text{Tr}(H_{D_{24}}) \approx -5.095 \times 10^{-3}.$$

These are full-crystal displacement results and are therefore logically distinct from the positive spherical-shell spectrum of $24A_1$.

8. MACROSCOPIC BORN ELASTICITY

The preceding shell and displacement Hessians should also be distinguished from continuum elasticity.

Let

$$\varepsilon \in \text{Sym}(24, \mathbb{R})$$

be a homogeneous strain. Under the Cauchy–Born hypothesis, the microscopic strain energy is determined by the bond deformation

$$R \mapsto (I + \varepsilon)R.$$

At a zero-Cauchy-stress equilibrium,

$$\sigma^0 = 0,$$

the first-order prestress contribution cancels.

For an admissible central potential

$$V(r) = f(r^2),$$

with

$$f''(u) > 0$$

at equilibrium, the quadratic strain energy takes the exact form

$$U(\varepsilon) = \frac{1}{v_c} \sum_{R \in \Lambda \setminus \{0\}} f''(\|R\|^2) (R^T \varepsilon R)^2 \geq 0. \quad (17)$$

Theorem 8.1 (Conditional Born elastic positivity). *At zero Cauchy stress, for every admissible potential satisfying*

$$f'' > 0,$$

the microscopic Cauchy–Born strain energy is an exact sum of non-negative squares. Consequently,

$$C \succ 0$$

for all twenty-four rank-24 Niemeier crystals under the hypotheses of the theorem.

Thus the spherical-shell instability does not imply negative macroscopic elastic moduli.

This distinction is essential:

spherical shell Hessian $\neq$ full-lattice displacement Hessian $\neq$ zero-stress Born elasticity.
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The three calculations probe different physical and geometric questions.

9. ELASTIC INVARIANT THEORY AND COXETER COLLISIONS

The point-group invariant spaces of the continuum elasticity tensor provide information not visible in scalar genus-one theta series.

Let

$$d_{\text{elast}}(R) = \dim \left(\text{Sym}^2(\text{Sym}^2(\mathbb{R}^{24})) \right)^{\text{Aut}(N(R))}$$

and

$$d_{\text{Cauchy}}(R) = \dim \left(\text{Sym}^4(\mathbb{R}^{24}) \right)^{\text{Aut}(N(R))}.$$

The Leech lattice is uniquely isotropic in the relevant sense:

$$d_{\text{elast}}(\Lambda_{24}) = 2, \quad d_{\text{Cauchy}}(\Lambda_{24}) = 1.$$

The $24A_1$ crystal has cubic symmetry in dimension 24 and

$$d_{\text{elast}}(24A_1) = 3, \quad d_{\text{Cauchy}}(24A_1) = 2.$$

More generally, all rooted crystals admit directional anisotropy.

Theorem 9.1 (Elastic separation). *For the five Coxeter collision pairs*

$$h \in \{6, 10, 12, 18, 30\},$$

the Cauchy-reduced invariant dimensions are distinct for the two lattices in each collision pair:

$$d_{\text{Cauchy}}(R_1) \neq d_{\text{Cauchy}}(R_2).$$

Hence macroscopic continuum elasticity separates all five Coxeter collision classes already at genus-one geometric order.

This result is complementary to scalar Siegel-theta separation: the two theories use different invariants and therefore have different minimal separation mechanisms.

10. GENUS-ONE THETA SERIES

The genus-one theta series of a Niemeier lattice is a modular form of weight 12.

Using the standard normalization of E_4^3 and Δ ,

$$\Theta_N^{(1)}(\tau) = E_4(\tau)^3 + (24h(N) - 720)\Delta(\tau). \quad (18)$$

Indeed,

$$E_4^3 = 1 + 720q + \dots, \quad \Delta = q + \dots,$$

so the coefficient of q in (18) is

$$24h(N),$$

the number of roots.

Consequently, different Coxeter numbers produce different genus-one theta series.

If desired, in the basis $\{E_{12}, \Delta\}$ one may equivalently write

$$\Theta_N^{(1)} = E_{12} + \left(24h(N) - \frac{65520}{691}\right)\Delta. \quad (19)$$

11. GENUS TWO AND THREE: COXETER RIGIDITY

The remarkable phenomenon occurs for the five pairs of distinct Niemeier lattices sharing the same Coxeter number.

For genus two,

$$a(I_2, \Theta_N^{(2)}) = 480h^2 + 144h. \quad (20)$$

For genus three,

$$a(I_3, \Theta_N^{(3)}) = 7872h^3 + 7104h^2 + 2880h \quad (21)$$

or equivalently

$$a(I_3, \Theta_N^{(3)}) = 192h(41h^2 + 37h + 15). \quad (22)$$

Thus for

$$g \leq 3$$

the scalar Siegel theta series depend only on h .

The five collision pairs are

h	Niemeier root systems
6	$A_5^{\oplus 4} \oplus D_4$ vs. $D_4^{\oplus 6}$
10	$A_9^{\oplus 2} \oplus D_6$ vs. $D_6^{\oplus 4}$
12	$A_{11} \oplus D_7 \oplus E_6$ vs. $E_6^{\oplus 4}$
18	$A_{17} \oplus E_7$ vs. $D_{10} \oplus E_7^{\oplus 2}$
30	$D_{16} \oplus E_8$ vs. $E_8^{\oplus 3}$.

For each pair,

$$\Theta_{N_1}^{(g)} = \Theta_{N_2}^{(g)}, \quad g = 1, 2, 3.$$

Hence

$$g_{\text{sep}} \geq 4. \quad (23)$$

12. SHARP GENUS-4 SEPARATION

Let

$$I_4$$

denote the 4×4 identity Gram matrix.

For an even lattice L , the Fourier coefficient

$$a(I_4, \Theta_L^{(4)})$$

counts ordered quadruples

$$(x_1, x_2, x_3, x_4)$$

satisfying

$$\langle x_i, x_j \rangle = 2\delta_{ij}.$$

Thus $a(I_4)$ counts ordered orthogonal 4-frames of roots.

Theorem 12.1 (Sharp Niemeier separation at genus four). *The coefficient*

$$a(I_4, \Theta_N^{(4)})$$

takes twenty-four pairwise distinct values as N ranges over all twenty-four Niemeier lattices.

Consequently,

$$\boxed{g_{\text{sep}} = 4.} \quad (24)$$

Thus genus four is not merely sufficient but strictly minimal for pairwise separation by the ordinary scalar Siegel theta series.

The sequence of implications is therefore

$$\boxed{\text{Coxeter rigidity} \longrightarrow \text{five collision classes} \longrightarrow a(I_4) \text{ separation} \longrightarrow g_{\text{sep}} = 4.}$$

13. THE SCHOTTKY FORM AND THE TORELLI LOCUS

Let

$$\mathcal{A}_4 = \text{Sp}(8, \mathbb{Z}) \backslash \mathbb{H}_4$$

be the Siegel modular variety and let

$$\mathcal{J}_4 \subset \mathcal{A}_4$$

denote the Torelli locus.

The Schottky form is

$$J_8(\Omega) = r_8(\Omega)^2 - 2^8 r_{16}(\Omega), \quad (25)$$

a weight-8 Siegel modular form satisfying

$$J_8|_{\mathcal{J}_4} = 0.$$

Multiplication by the weight-4 Eisenstein series gives

$$F_{12} = J_8 E_4, \quad (26)$$

which lies in the weight-12 cusp space and vanishes on the Jacobian locus.

The relevant Schottky-vanishing subspace is one-dimensional.

For a Niemeier lattice N , define

$$\Delta\Theta_N^{(4)} = \Theta_N^{(4)} - \Theta_{\Lambda_{24}}^{(4)}. \quad (27)$$

The collision differences at fixed Coxeter number lie in the one-dimensional Schottky direction.

Theorem 13.1 (Schottky collapse). *For each of the five equal-Coxeter collision classes, the genus-4 theta-series difference is proportional to the Schottky direction:*

$$F_h = \Theta_{N_{h,1}}^{(4)} - \Theta_{N_{h,2}}^{(4)} \in \mathbb{C} (J_8 \Theta_{E_8}^{(4)}). \quad (28)$$

Consequently, these differences vanish after restriction to the Jacobian locus $\mathcal{J}_4$.

Thus the ambient genus-4 separation has a nontrivial geometric collapse on the Torelli locus.

14. COXETER AMPLITUDE OF THE SCHOTTKY LIFT

The root count satisfies

$$|R(N)| = 24h(N).$$

The certified Schottky/transverse calculation identifies the corresponding lift amplitude, relative to the Leech anchor, with a quantity proportional to

$$h(N)|R(N)| = 24h(N)^2. \quad (29)$$

Thus

$$\langle \Theta_N^{(4)} - \Theta_{\Lambda_{24}}^{(4)}, F_{12} \rangle \propto 24h(N)^2. \quad (30)$$

The proportionality constant depends on the normalization of the Schottky form and Petersson pairing.

For the Leech lattice,

$$h(\Lambda_{24}) = 0,$$

so

$$\langle \Theta_{\Lambda_{24}}^{(4)} - \Theta_{\Lambda_{24}}^{(4)}, F_{12} \rangle = 0.$$

The Leech lattice is therefore the natural rootless zero-amplitude anchor of this Schottky comparison.

It is important not to confuse the absolute amplitude

$$24h^2$$

with the genus-1 theta coefficient, which is linear in h through the root count $24h$.

15. LEECH UNIVERSAL OPTIMALITY

The Leech lattice has no roots:

$$R(\Lambda_{24}) = \emptyset.$$

Its minimal vectors have norm squared 4 and number

$$196560.$$

The Leech lattice is universally optimal among periodic configurations of density one in $\mathbb{R}^{24}$ for completely monotonic radial potentials.

In particular, for the Gaussian interaction,

$$E_\alpha(L) = \frac{1}{2} \sum_{v \neq 0} e^{-\pi\alpha\|v\|^2},$$

the Leech lattice is a global minimizer.

The exact Riesz-shell and full-lattice spectral calculations give, in the appropriate normalization, the positive ground-state value

$$\lambda_{\text{ground}} = \frac{73073}{58982400} > 0. \quad (31)$$

The corresponding single-particle stiffness is

$$\lambda_S = \frac{1200199}{196608000} > 0, \quad (32)$$

with collective screening ratio

$$\kappa = \frac{730}{3597}. \quad (33)$$

These exact positive values are consistent with the universal optimality of the rootless Leech configuration.

16. THE THREE STABILITY NOTIONS

The preceding results can now be organized without conflation.

Object	Interaction / constraint	Principal conclusion
Riemannian modular Hessian	Gaussian lattice energy under e^{tX} , $X \in \text{Sym}_0(24, \mathbb{R})$	Exact formula (8). Sign depends on the competition between longitudinal and pre-stress terms.
Minimal spherical root shell	$F(U) = U^{-2}$ on $S_{\sqrt{2}}^{23}$	$24A_1$ is the unique stable rooted shell; the other 22 rooted shells are tachyonic.
Full periodic displacement Hessian	$F(U) = U^{-2}$	Every rooted Niemeier lattice is a Morse saddle; the Leech lattice has a positive isolated ground state.
Continuum Born elasticity	Zero Cauchy stress and $f'' > 0$	Exact sum-of-squares positivity; $C \succ 0$ for all twenty-four crystals.
Scalar Siegel theta series	Genus g	Coxeter rigidity for $g \leq 3$; complete separation first occurs at $g = 4$.
Schottky restriction	Genus 4, restriction to $\mathcal{J}_4$	Collision differences collapse onto the one-dimensional Schottky direction.

TABLE 1. Distinct geometric and modular stability/separation mechanisms.

17. REPRESENTATIVE NIEMEIER INVARIANTS

18. DISCUSSION

The central conclusion is a hierarchy rather than a single universal Hessian.

First, the rank-24 root systems carry a common Coxeter invariant. At genus one this invariant controls the root multiplicity:

$$|R| = 24h.$$

Second, the scalar Siegel theta series remain rigid within Coxeter collision classes through genus three:

$$\Theta_{N_1}^{(g)} = \Theta_{N_2}^{(g)}, \quad g \leq 3,$$

Lattice	h	$ R $	$24h^2$	Root-shell status
Λ_{24}	0	0	0	Rootless
$24A_1$	2	48	96	Unique stable rooted shell
$12A_2$	3	72	216	Tachyonic
$6D_4$	6	144	864	Tachyonic
$4E_6$	12	288	3456	Tachyonic
$3E_8$	30	720	21600	Tachyonic
D_{24}	46	1104	50784	Tachyonic

TABLE 2. Representative Coxeter, root-count, Schottky-amplitude, and minimal-shell invariants.

whenever

$$h(N_1) = h(N_2).$$

Third, genus four introduces genuinely new frame information. The coefficient

$$a(I_4, \Theta_N^{(4)})$$

counts orthogonal root 4-frames and separates all twenty-four Niemeier lattices.

Fourth, the same transition admits a geometric interpretation, but only after keeping the relevant Hessians distinct. The spherical minimal-shell problem selects $24A_1$ as the unique stable rooted shell, whereas the full periodic displacement problem produces a negative transverse mode for every rooted Niemeier lattice. The continuum zero-stress Born theory is again different and is positive for all twenty-four crystals.

Finally, the rootless Leech lattice is singled out simultaneously by the absence of a root shell, the positive full-lattice spectral gap, universal optimality, and the vanishing Schottky amplitude.

The resulting architecture may be summarized as

$$\begin{array}{c}
 \text{Coxeter rigidity} \\
 \downarrow \\
 g \leq 3 : \text{ scalar theta collision classes} \\
 \downarrow \\
 g = 4 : a(I_4) \text{ separates all 24} \\
 \downarrow \\
 \text{Schottky/Torelli collapse} \\
 \parallel \\
 \text{distinct lattice spectral and elastic mechanisms}
 \end{array} \tag{34}$$

No identification of these mechanisms is required for the mathematical statements above. Their common feature is instead that the same rank-24 arithmetic data become visible through different invariant functors: root multiplicities, orthogonal-frame counts, discriminant glue, spherical spectra, continuum elasticity, and Siegel modular forms.

19. CONCLUSION

We have assembled a consistent rank-24 framework with four separate levels.

- (1) The Riemannian modular Hessian on

$$\mathrm{SL}(24, \mathbb{R})/\mathrm{SO}(24)$$

admits the exact Gaussian second-variation formula (8).

- (2) The canonical Riesz spherical-shell problem has an exact stability dichotomy: $24A_1$ is the unique stable rooted shell, while the other twenty-two rooted shells are tachyonic.
- (3) The full periodic displacement Hessian exhibits universal transverse instability for all twenty-three rooted Niemeier lattices, whereas the Leech lattice possesses a positive isolated spectral ground state. At zero Cauchy stress, however, the macroscopic Born elastic tensor is positive definite under the stated convexity assumptions.
- (4) The scalar Siegel theta series have sharp separation genus

$$g_{\mathrm{sep}} = 4,$$

with the single coefficient $a(I_4)$ providing global injectivity on the twenty-four Niemeier classes. The equal-Coxeter collision differences lie in the Schottky direction, with the corresponding absolute lift amplitude proportional to

$$24h^2.$$

The Leech lattice therefore occupies a distinguished position across the different theories, but the distinctions between shell stability, periodic displacement stability, continuum elasticity, and modular separation are essential to the mathematical integrity of the framework.

REFERENCES

- [1] H.-V. Niemeier, *Definite quadratische Formen der Dimension 24 und Diskriminante 1*, J. Number Theory **5** (1973), 142–178.
- [2] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, 1999.
- [3] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska, *Universal optimality of the E_8 and Leech lattices and interpolation formulas*, Ann. of Math. **196** (2022), 983–1082.
- [4] V. A. Erokhin, *Theta-series of even unimodular 24-dimensional lattices*, Zap. Nauchn. Sem. LOMI **116** (1982), 44–55.
- [5] G. Nebe and B. Venkov, *The Niemeier lattices, genus 3 theta series, and harmonic polynomials*, J. Reine Angew. Math. **538** (2001), 49–68.
- [6] B. B. Venkov, *Even unimodular extremal lattices*, Proc. Steklov Inst. Math. **165** (1984), 47–52.
- [7] SRFP311T1 Collaboration, *Spectral Geometry of Niemeier Root Shells, Discriminant Group Algebras, and the 24-Dimensional Landscape*, September 2026.
- [8] SRFP311T1 Collaboration, *Sharp Niemeier Theta Separation at Genus Four*, September 2026.
- [9] SRFP311T1 Collaboration, *Universal Transverse Instability of Niemeier Root Lattices*, September 2026.
- [10] SRFP311T1 Collaboration, *Niemeier Elastic Classification, Acoustic Birefringence & Born Stability*, September 2026.