

PRUNED LEECH CUBATURES AND MODULAR CONE GEOMETRY: SPARSE SUPPORTS, EXCLUSION GAPS, AND NUMERICAL EVIDENCE FOR CUSP-FORM SIGN VARIATION

SRFP311T1 COLLABORATION

ABSTRACT. We investigate sparse positive-weight spherical cubature constructions supported on concentric shells of the 24-dimensional Leech lattice. The modular-form reduction of Venkov identifies the relevant harmonic conditions with Fourier-coefficient conditions in spaces of cusp forms. This permits the cubature problem to be formulated as a finite-dimensional non-negative linear feasibility problem.

We distinguish carefully between the modular dimension at a single harmonic degree and the cumulative dimension of the direct-sum evaluation space obtained by imposing all relevant even harmonic conditions up to a target degree.

Using a sparse non-negative least-squares formulation, we obtain numerical candidates for positive cubatures at degrees $k = 114$ and $k = 120$, corresponding to spherical strengths 115 and 121, respectively. For $k = 114$ the computation uses a candidate pool $\{12, \dots, 265\}$ and produces a support of 54 shells, with the second active shell at $m = 129$. For $k = 120$ the candidate pool $\{12, \dots, 295\}$ produces a support of 53 shells, with the second active shell at $m = 89$. Thus both computations exhibit pronounced internal exclusion gaps.

We interpret these observations as numerical evidence for sparse-support phenomena in the modular evaluation cone. We also formulate a conjectural asymptotic description of the location of an exclusion boundary. The proposed scale is $m \asymp k^2/96$, but we do not claim a theorem establishing this law.

A separate finite-field computation gives full column rank for the 271×53 integer Fourier-coefficient matrix associated with the reported $k = 120$ support. This establishes linear independence of the selected generators, provided the stated exact matrix and modular calculation are verified. It does not by itself establish that the generated cone is a face of the full modular cone or that every cusp form changes sign on the selected support.

Finally, we state the exact convex-duality criterion that would convert an exact strictly positive annihilating vector into a finite-support sign-variation theorem. The numerical residuals reported here provide computational evidence for approximate annihilation, but should not be confused with exact rational or interval certificates.

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1. INTRODUCTION

The interaction between spherical designs, Euclidean lattices, and modular forms provides a particularly rigid setting in which high-dimensional cubature problems can be reduced to finite-dimensional linear algebra.

Let $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ be the unit sphere equipped with normalized rotation-invariant measure σ .

Definition 1.1 (Weighted spherical design). A finite set $X \subset \mathbb{S}^{d-1}$ with positive weights $W : X \rightarrow \mathbb{R}_{>0}$ is a weighted spherical t -design if

$$\int_{\mathbb{S}^{d-1}} f(\xi) d\sigma(\xi) = \frac{\sum_{x \in X} W(x) f(x)}{\sum_{x \in X} W(x)} \quad (1.1)$$

for every polynomial f of total degree at most t .

By harmonic decomposition, this is equivalent to the vanishing of all nonconstant harmonic moments through degree t :

$$\sum_{x \in X} W(x) P_j(x) = 0, \quad P_j \in \text{Harm}_j(\mathbb{R}^d), \quad 1 \leq j \leq t. \quad (1.2)$$

For the antipodal shell-supported constructions considered here, the odd-degree harmonic conditions vanish automatically. Thus the nontrivial conditions arise from the even degrees.

1.1. The Leech lattice. Let $\Lambda_{24} \subset \mathbb{R}^{24}$ denote the Leech lattice. For $m \geq 2$, define the shell

$$S_m = \{x \in \Lambda_{24} : \|x\|^2 = 2m\}. \quad (1.3)$$

The minimal shell is S_2 , consisting of 196560 vectors. After radial projection to $\mathbb{S}^{23}$, this shell forms a spherical 11-design.

The theta series of the Leech lattice is

$$\Theta_{\Lambda_{24}}(\tau) = \sum_{x \in \Lambda_{24}} q^{\|x\|^2/2} = 1 + \sum_{m=2}^{\infty} N_m q^m, \quad q = e^{2\pi i \tau}. \quad (1.4)$$

Since $\Theta_{\Lambda_{24}}$ is a modular form of weight 12 and has no q -term, one has

$$\Theta_{\Lambda_{24}} = E_{12} - \frac{65520}{691}\Delta, \quad (1.5)$$

and therefore

$$N_m = \frac{65520}{691} (\sigma_{11}(m) - \tau(m)). \quad (1.6)$$

2. HARMONIC THETA SERIES AND MODULAR REDUCTION

Let $P_j \in \text{Harm}_j(\mathbb{R}^{24})$ be homogeneous harmonic of even degree j . Its harmonic theta series is

$$\Theta_{\Lambda_{24}, P_j}(\tau) = \sum_{x \in \Lambda_{24}} P_j(x) q^{\|x\|^2/2}. \quad (2.1)$$

For $j > 0$, this is a cusp form of weight $j + 12$.

Because the Leech lattice has no vectors of norm 2, the harmonic theta series has no constant or q^1 term. Consequently,

$$\Theta_{\Lambda_{24}, P_j} \in S_{j+12}^{(2)}(\text{SL}_2(\mathbb{Z})), \quad (2.2)$$

where

$$S_w^{(2)}(\text{SL}_2(\mathbb{Z})) = \{f \in S_w(\text{SL}_2(\mathbb{Z})) : \text{ord}_q(f) \geq 2\}. \quad (2.3)$$

The automorphism group Co_0 acts on harmonic polynomials. For P_j orthogonal to the invariant subspace, its shell sums vanish.

Proposition 2.1 (Reynolds annihilation). *Let $G = \text{Co}_0$. If*

$$P_j \perp (\text{Harm}_j(\mathbb{R}^{24}))^G,$$

then

$$\sum_{x \in S_m} P_j(x) = 0$$

for every $m \geq 2$.

Proof. For every shell S_m ,

$$\sum_{x \in S_m} P_j(x) = \sum_{x \in S_m} \frac{1}{|G|} \sum_{g \in G} P_j(gx).$$

The inner average is the Reynolds projection of P_j onto $(\text{Harm}_j)^G$. The orthogonality assumption therefore makes the projection zero. $\square$

For the invariant subspace, Venkov's harmonic theta construction gives the relevant modular-form model.

Theorem 2.2 (Venkov). *For the degrees considered in this paper, the harmonic theta map identifies the relevant Co_0 -invariant harmonic subspace with the corresponding space of weight- $j + 12$ cusp forms vanishing to order at least 2. Thus, in the invariant sector,*

$$(\text{Harm}_j(\mathbb{R}^{24}))^{\text{Co}_0} \cong S_{j+12}^{(2)}(\text{SL}_2(\mathbb{Z})).$$

Remark 2.3. The precise range and normalization of the preceding identification should be understood in the sense of the harmonic theta-series construction of Venkov. In a final version, the exact theorem or proposition reference should be supplied.

Since every element of $S_w^{(2)}$ is divisible by Δ^2 , division by Δ^2 gives an isomorphism

$$S_{j+12}^{(2)}(\mathrm{SL}_2(\mathbb{Z})) \xrightarrow{\sim} M_{j-12}(\mathrm{SL}_2(\mathbb{Z})), \quad f \mapsto \frac{f}{\Delta^2}. \quad (2.4)$$

We therefore define the *single-degree modular dimension*

$$d_j = \dim M_{j-12}(\mathrm{SL}_2(\mathbb{Z})). \quad (2.5)$$

For the two target degrees considered below,

$$d_{114} = \dim M_{102}(\mathrm{SL}_2(\mathbb{Z})) = 9, \quad d_{120} = \dim M_{108}(\mathrm{SL}_2(\mathbb{Z})) = 10. \quad (2.6)$$

These numbers describe the modular-form dimension at the individual top degree. They are not the dimensions of the cumulative evaluation spaces used when all intermediate even harmonic conditions are imposed.

3. THE CUMULATIVE MODULAR EVALUATION SPACE

A spherical design of strength t must satisfy harmonic conditions at every degree $j \leq t$. For the antipodal shell-supported constructions considered here, the relevant degrees are the even degrees beginning at 12.

For an even target degree $k \geq 12$, define the cumulative modular space

$$\mathcal{M}_{\leq k} = \bigoplus_{\substack{12 \leq j \leq k \\ j \text{ even}}} S_{j+12}^{(2)}(\mathrm{SL}_2(\mathbb{Z})). \quad (3.1)$$

Its dimension is

$$D_k = \dim \mathcal{M}_{\leq k} = \sum_{\substack{12 \leq j \leq k \\ j \text{ even}}} d_j. \quad (3.2)$$

For the two target degrees,

$$D_{114} = 243, \quad D_{120} = 271. \quad (3.3)$$

Thus the values 9 and 10 refer to the top-degree spaces, whereas 243 and 271 refer to the cumulative direct-sum spaces.

This distinction is essential for interpreting the computational matrices.

Choose, for each even j with $12 \leq j \leq k$, a basis

$$F_{j,1}, \dots, F_{j,d_j}$$

of $S_{j+12}^{(2)}(\mathrm{SL}_2(\mathbb{Z}))$.

For a shell index m , define the degree- j evaluation vector

$$v_j(m) = (2m)^{-j/2} (a_m(F_{j,1}), \dots, a_m(F_{j,d_j}))^T. \quad (3.4)$$

The cumulative evaluation vector is then

$$v_{\leq k}(m) = \bigoplus_{\substack{12 \leq j \leq k \\ j \text{ even}}} v_j(m) \in \mathbb{R}^{D_k}. \quad (3.5)$$

For a finite shell set $\mathcal{S}$, the cumulative evaluation matrix is

$$A_{\leq k, \mathcal{S}} = (v_{\leq k}(m))_{m \in \mathcal{S}} \in \mathbb{R}^{D_k \times |\mathcal{S}|}. \quad (3.6)$$

A positive shell-weight vector

$$W = (W_m)_{m \in \mathcal{S}} \in \mathbb{R}_{>0}^{|\mathcal{S}|}$$

annihilates all relevant harmonic conditions through degree k precisely when

$$A_{\leq k, \mathcal{S}} W = 0. \quad (3.7)$$

4. THE MODULAR EVALUATION CONE

Fix an even degree $k \geq 12$. The modular evaluation cone generated by a finite shell set $\mathcal{S}$ is

$$\mathcal{C}_{\leq k}(\mathcal{S}) = \text{cone}\{v_{\leq k}(m) : m \in \mathcal{S}\}. \quad (4.1)$$

The cubature problem is therefore a finite-dimensional positive dependence problem in this cone.

A positive cubature supported on $\mathcal{S}$ corresponds to

$$W \in \mathbb{R}_{>0}^{|\mathcal{S}|}$$

such that

$$A_{\leq k, \mathcal{S}} W = 0.$$

Equivalently, the origin lies in the relative interior of the convex cone generated by the selected evaluation vectors, subject to the chosen normalization.

4.1. Gordan's alternative. The following convex-duality statement is the finite-dimensional tool needed for the sign-variation discussion.

Theorem 4.1 (Gordan's alternative). *Let*

$$A \in \mathbb{R}^{C \times n}.$$

Exactly one of the following alternatives holds:

(a) *there exists $W \in \mathbb{R}_{>0}^n$ such that*

$$AW = 0;$$

(b) *there exists $c \in \mathbb{R}^C \setminus \{0\}$ such that*

$$A^T c \geq 0.$$

Corollary 4.2 (Conditional sign variation). *Suppose there exists an exact vector*

$$W \in \mathbb{R}_{>0}^n$$

satisfying

$$AW = 0.$$

Assume additionally that

$$A^T c = 0 \implies c = 0.$$

Then every nonzero c has both a positive and a negative coordinate in $A^T c$.

Proof. Suppose that

$$A^T c \geq 0.$$

Then

$$0 = c^T AW = (A^T c)^T W.$$

Since every coordinate of W is strictly positive and every coordinate of $A^T c$ is nonnegative, it follows that

$$A^T c = 0.$$

The injectivity hypothesis then gives $c = 0$. Thus a nonzero c cannot have all nonnegative evaluations. Applying the same argument to $-c$ rules out all nonpositive evaluations. $\square$

Remark 4.3. The injectivity hypothesis is logically independent of full column rank. If A is a $C \times n$ matrix with $n < C$, then

$$\dim \ker(A^T) \geq C - n.$$

Thus full column rank of the selected shell vectors does not by itself imply injectivity of the evaluation map on the full ambient space.

5. SPARSE NON-NEGATIVE LEAST SQUARES

Let

$$\mathcal{D} = \{s_{\min}, \dots, s_{\max}\}$$

be a candidate shell pool, and choose an anchor shell $s_0 \in \mathcal{D}$. Normalize its weight to 1 and solve

$$\min_{W_{\text{rest}} \geq 0} \|A_{\leq k, \mathcal{D} \setminus \{s_0\}} W_{\text{rest}} + v_{\leq k}(s_0)\|_2^2. \quad (5.1)$$

This is a convex non-negative least-squares problem.

If the minimum is exactly zero, the resulting vector gives an exact non-negative annihilation certificate. If the computed minimum is merely small, the result is instead a numerical approximate annihilator.

For reproducibility, define the active support using the explicit numerical threshold

$$\varepsilon_{\text{act}} = 10^{-12}.$$

Thus

$$\mathcal{S}_{\text{active}} = \{m \in \mathcal{D} : W_m > \varepsilon_{\text{act}}\}. \quad (5.2)$$

The threshold is a computational convention, not a mathematical invariant. Consequently, support counts should be accompanied by stability tests under changes in precision, solver tolerances, and activation threshold.

6. NUMERICAL EXCLUSION GAPS

The computations reported below were performed using the cumulative modular evaluation matrices described in Sections 3 and 4. The reported residuals are floating-point quantities.

Definition 6.1 (Numerical exclusion gap). Let

$$\mathcal{S}_{\text{active}} = \{m_1 < m_2 < \dots < m_r\}$$

be the set of indices satisfying (5.2). An exclusion gap is an interval

$$m_i < m < m_{i+1}$$

containing no active shell.

6.1. Degree $k = 114$. For

$$k = 114,$$

we used the candidate pool

$$\mathcal{D} = \{12, \dots, 265\}.$$

The numerical solution contains 54 active shells under the threshold $\varepsilon_{\text{act}} = 10^{-12}$. The first active shell is $m = 12$, while the second active shell is $m = 129$.

Thus the observed first internal gap has index width

$$129 - 12 = 117,$$

corresponding to the empty shell indices

$$13, 14, \dots, 128.$$

Representative weights from the numerical solution are

$$\begin{aligned} W_{12} &\approx 1.000000 \times 10^0, \\ W_{129} &\approx 1.170522 \times 10^{-7}, \\ W_{153} &\approx 2.860106 \times 10^{-8}, \\ W_{265} &\approx 9.316376 \times 10^{-9}. \end{aligned}$$

These values should be regarded as floating-point numerical data rather than exact weights.

6.2. **Degree $k = 120$.** For

$$k = 120,$$

we used

$$\mathcal{D} = \{12, \dots, 295\}.$$

The reported numerical solution contains 53 active shells. The beginning of the support is

$$12, \quad 89, \quad 150, \quad 168, \dots$$

Consequently, the first exclusion gap has index width

$$89 - 12 = 77,$$

corresponding to the empty shell indices

$$13, \dots, 88.$$

This differs from the first gap observed at $k = 114$ and demonstrates that the support pattern is degree-dependent.

7. A CONJECTURAL HORIZON FOR THE EXCLUSION GAP

The numerical experiments motivate a possible relationship between the degree k and the location of an exclusion boundary.

A heuristic scale considered in the underlying computational study is

$$m_*(k) \asymp \frac{k^2}{96}. \tag{7.1}$$

We stress that the present manuscript does not prove this asymptotic law.

Conjecture 7.1 (Spectral horizon). *There exists a constant or slowly varying correction term such that a stable high-frequency support region for degree k occurs on the scale*

$$m_*(k) = \frac{k^2}{96} + o(k^2).$$

A sharper conjectural form suggested by the computational heuristic is

$$m_*(k) \approx \frac{k^2}{96} - \frac{k}{12} + O(1).$$

For $k = 114$,

$$\frac{114^2}{96} = 135.375,$$

while the observed second active shell is 129.

For $k = 120$,

$$\frac{120^2}{96} = 150,$$

while the observed second active shell is 89.

The two reported computations therefore do not by themselves constitute evidence for a monotone realization of the proposed quadratic scale. They are better viewed as motivating data for a broader computational test.

Additional computations over a substantially wider range of degrees, candidate pools, precisions, and support-selection procedures would be required before drawing an asymptotic conclusion.

Remark 7.2. The language of “Eisenstein coherence,” “Deligne oscillation,” and “phase locking” is presently heuristic. A theorem connecting these asymptotic mechanisms to the geometry of the positive cone would require quantitative estimates on the relevant Fourier-coefficient vectors and their angular distribution.

8. RANK AND SPARSE SUPPORT

For $k = 120$, the active shell index set is

$$\begin{aligned} \mathcal{S}_{120} = \{ & 12, 89, 150, 168, 177, 178, 180, 189, 196, 200, 208, 212, 213, 214, \\ & 215, 218, 220, 222, 227, 230, 235, 236, 242, 243, 246, 248, 250, \\ & 252, 253, 256, 257, 258, 262, 267, 269, 271, 272, 273, 274, 276, \\ & 277, 281, 282, 283, 285, 286, 287, 288, 289, 290, 291, 292, 295 \}. \end{aligned} \quad (8.1)$$

This set contains 53 shell indices.

Choose the normalized integral basis

$$F_{a,b} = \Delta^2 E_4^a E_6^b, \quad 4a + 6b = j - 12, \quad (8.2)$$

for each relevant even degree j , with the standard normalized Eisenstein-series conventions

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

and

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Under these conventions, every Fourier coefficient of $F_{a,b}$ is an integer.

For the cumulative degree-120 space, assemble all such basis elements for

$$j = 12, 14, \dots, 120.$$

Define the exact integer coefficient matrix

$$B_{\text{active}} = (a_m(F))_{\substack{F \in \mathcal{B}_{\leq 120} \\ m \in \mathcal{S}_{120}}} \in \mathbb{Z}^{271 \times 53}, \quad (8.3)$$

where $\mathcal{B}_{\leq 120}$ is the chosen integral basis of the cumulative modular space

$$\mathcal{M}_{\leq 120}.$$

This matrix is deliberately distinguished from the real evaluation matrix $A_{\leq 120, \mathcal{S}_{120}}$, whose columns contain the additional radial factors $(2m)^{-j/2}$ degree by degree.

8.1. **Finite-field rank.** Let

$$p = 2^{31} - 1 = 2\,147\,483\,647.$$

The exact integer matrix (8.3) can be reduced entrywise modulo p . The reported computation gives

$$\text{rank}_{\mathbb{F}_p}(B_{\text{active}}) = 53. \quad (8.4)$$

Proposition 8.1 (Independence of the selected generators). *Suppose the modular computation establishes a nonzero 53×53 minor of B_{active} modulo p . Then the selected columns of B_{active} are linearly independent over $\mathbb{Q}$, and hence over $\mathbb{R}$.*

Proof. Let M be the corresponding 53×53 minor. By assumption,

$$M \not\equiv 0 \pmod{p}.$$

Since M has integer entries, its determinant is an integer that is not divisible by p . In particular,

$$\det(M) \neq 0$$

over $\mathbb{Z}$, and hence over $\mathbb{Q}$ and $\mathbb{R}$. Therefore the selected columns are linearly independent. $\square$

Remark 8.2. This is an exact rank statement conditional only on the correctness of the reported finite-field computation. It does not prove that the selected generators form a face of the full modular cone.

To prove a face property, one would additionally need a supporting functional that separates the selected generators from all non-selected generators.

9. NUMERICAL CUBATURE DATA

The following table summarizes the computations discussed above.

TABLE 1. Numerical sparse-support computations compared with contiguous baselines.

k	Strength t	D_k	Method	Active shells	First gap	Rel. residual	Status
110	111	225	Contiguous	226	0	Exact	Prev. exact
114	115	243	Contiguous	244	–	Infeasible	Prev. fail
114	115	243	Sparse NNLS	54	117	1.84×10^{-15}	Numerical
120	121	271	Sparse NNLS	53	77	5.92×10^{-16}	Numerical

Here D_k denotes the dimension of the cumulative modular evaluation space defined in Section 3. It should not be confused with the top-degree dimension d_k .

The residual values in Table 1 are floating-point numerical residuals. They do not constitute exact proofs that the corresponding modular equations vanish.

9.1. **Point counts.** The total number of lattice points represented by a shell-supported cubature is

$$N_{\text{total}} = \sum_{m \in \mathcal{S}_{\text{active}}} N_m. \quad (9.1)$$

Using (1.6), this quantity can be evaluated exactly once the support is fixed.

Since

$$N_m \asymp m^{11},$$

the shell cardinality has polynomial order m^{11} . Consequently, a comparison of total point counts should use the complete shell list rather than only the largest shell index.

For the $k = 120$ support, the relevant exact quantity is

$$N_{\text{total}}(120) = \sum_{m \in \mathcal{S}_{120}} \frac{65520}{691} (\sigma_{11}(m) - \tau(m)).$$

An explicit integer evaluation should be included once the computational data are independently verified.

10. SIGN VARIATION: WHAT REMAINS TO BE PROVED

The convex-duality argument gives a clean route from an exact positive cubature to sign variation, but the required hypotheses must be stated carefully.

Let

$$A_{\text{active}} \in \mathbb{R}^{D_k \times r}$$

be an exact cumulative evaluation matrix for a support of r shells.

Suppose there exists

$$W \in \mathbb{R}_{>0}^r$$

such that

$$A_{\text{active}} W = 0.$$

Then Gordan's theorem rules out a nonzero vector c satisfying

$$A_{\text{active}}^T c \geq 0.$$

Thus every nonzero modular coordinate vector has at least one negative evaluation on the selected support. Applying the same argument to $-c$ gives a positive evaluation as well, provided the zero-evaluation kernel has been eliminated.

Proposition 10.1 (Conditional sign variation). *Let A_{active} be an exact evaluation matrix and suppose*

$$A_{\text{active}} W = 0$$

for some $W > 0$.

If the only modular form whose evaluations vanish at every selected shell is the zero form, then every nonzero modular form has both positive and negative evaluations on the selected support.

Proof. Suppose

$$A_{\text{active}}^T c \geq 0.$$

Then

$$0 = c^T A_{\text{active}} W = (A_{\text{active}}^T c)^T W.$$

Every coordinate of W is strictly positive, so

$$A_{\text{active}}^T c = 0.$$

By the assumed injectivity of the evaluation map, $c = 0$.

Therefore no nonzero modular form can have all nonnegative evaluations. Applying the same argument to $-c$ rules out all nonpositive evaluations. $\square$

Remark 10.2. The dimension count must be applied to the actual coordinate space being used.

For example, if the matrix is genuinely

$$271 \times 53,$$

then

$$A_{\text{active}}^T : \mathbb{R}^{271} \longrightarrow \mathbb{R}^{53}$$

has kernel dimension at least

$$271 - 53 = 218.$$

Consequently, full column rank of the 53 selected generators cannot establish injectivity on a 271-dimensional ambient coordinate space.

This is not a contradiction with the finite-field rank calculation: the latter concerns independence of the selected columns, whereas sign variation requires injectivity of the transpose evaluation map on the relevant modular-form space.

If the 271 coordinates are a faithful basis of the cumulative modular space, then the dimension obstruction is genuine. If a redundant coordinate realization is used, the injectivity question must instead be formulated on the corresponding quotient or basis space.

11. EXACT CERTIFICATION VERSUS FLOATING-POINT EVIDENCE

The distinction between numerical evidence and mathematical certification is central to the present work.

The residual

$$5.92 \times 10^{-16}$$

is evidence that a double-precision numerical solver found a vector close to the nullspace of the evaluation matrix. It does not prove that an exact null vector exists.

Several routes are available for upgrading the computation to an exact certificate.

- (i) **Exact rational reconstruction.** Given sufficiently accurate numerical weights, one may attempt to recover rational values and verify

$$AW = 0$$

using exact arithmetic.

- (ii) **Interval arithmetic.** One may compute rigorous enclosures for the residual and establish a certified bound.
- (iii) **Exact nullspace computation.** For sufficiently small support, rational linear algebra can determine whether the matrix has a positive null vector exactly.
- (iv) **Certified cone separation.** A rational dual vector can establish the relevant Gordan or Farkas alternative exactly.
- (v) **Higher-precision stability testing.** Repeating the NNLS computation at substantially higher precision and comparing support and weights can test whether the observed support is numerically stable, although stability alone is not an exact proof.

Until such a procedure is supplied, the appropriate terminology is “numerical candidate,” “approximate annihilator,” or “computational evidence,” rather than “unconditional proof.”

12. FINITE-FIELD VERIFICATION

Finite-field arithmetic is useful for detecting exact rank without the coefficient growth associated with fraction-free elimination over $\mathbb{Q}$.

Let

$$p = 2^{31} - 1 = 2\,147\,483\,647.$$

The matrix used for the rank computation is the exact integer matrix B_{active} defined in (8.3), not the radially scaled real matrix A_{active} .

The reported computation gives

$$\text{rank}_{\mathbb{F}_p}(B_{\text{active}}) = 53.$$

For a reproducible certificate, the accompanying computational data should include at least one of the following:

- (i) the complete finite-field row-reduction record;
- (ii) the 53 pivot rows and columns; or

(iii) a single 53×53 minor whose determinant is explicitly shown to be nonzero modulo p .
The third option gives a particularly compact certificate: if

$$\det M \not\equiv 0 \pmod{p},$$

then the corresponding integer determinant is nonzero.

Remark 12.1. The finite-field calculation and the floating-point NNLS calculation serve different purposes.

The finite-field calculation can provide an exact rank certificate for an integer matrix.

The NNLS calculation supplies numerical evidence for positivity and approximate annihilation of the selected shell weights.

Neither calculation alone proves the full conditional sign-variation statement.

13. STABILITY QUESTIONS AND THE INFINITE-DIMENSIONAL LIMIT

The finite-dimensional cone problem raises a natural question: whether families of finite positive certificates can converge to a meaningful positive functional in an infinite-dimensional setting.

Such a statement requires substantially more structure than the finite-dimensional Gordan alternative.

A future infinite-dimensional formulation would need to specify:

Condition 1: a precise topological vector space E of test functions;

Condition 2: a dual space E^* in which the limiting functionals live;

Condition 3: a topology in which bounded sequences of certificates possess convergent subsequences;

Condition 4: compatibility between finite-dimensional projections and the limiting functional;

Condition 5: closedness of the relevant positivity cone; and

Condition 6: a normalization preventing convergence to the zero functional.

These conditions constitute a framework for future analysis rather than a theorem in the absence of a fully specified functional-analytic model.

13.1. Observed growth of numerical weights. For illustration, the numerical calculations reported in the underlying computational study gave the following unnormalized ℓ^1 norms:

$$\begin{aligned} k = 12 : & \quad 1.24 \times 10^0, \\ k = 16 : & \quad 7.17 \times 10^0, \\ k = 20 : & \quad 5.29 \times 10^0, \\ k = 24 : & \quad 1.35 \times 10^1, \\ k = 28 : & \quad 2.49 \times 10^1, \\ k = 32 : & \quad 1.99 \times 10^2. \end{aligned}$$

These values suggest rapid growth in the normalization used by the computation. They do not, by themselves, establish exponential growth, nor do they prove failure of every alternative normalization or limiting procedure.

14. REPRODUCIBILITY REQUIREMENTS

To make the numerical claims independently verifiable, the following information should accompany the manuscript:

- (i) the exact definitions and normalizations of the modular bases;
- (ii) the complete candidate pools $\mathcal{D}$;
- (iii) the precise construction of the cumulative evaluation matrices;
- (iv) the NNLS solver and numerical precision;
- (v) stopping tolerances and conditioning diagnostics;
- (vi) the activation threshold ε_{act} ;
- (vii) the complete numerical weight vectors;
- (viii) the exact integer Fourier-coefficient matrices;
- (ix) the finite-field pivot data or a nonzero maximal minor;
- (x) independent residual calculations at higher precision;
- (xi) support-stability tables under threshold variation; and
- (xii) scripts sufficient to reproduce every table in the manuscript.

In particular, the number of active shells should be regarded as a reproducible computational statistic rather than a mathematical invariant until its stability under precision, tolerance, and threshold changes has been established.

15. DISCUSSION

The principal computational observation of this work is that sparse non-negative optimization produces shell supports very different from the contiguous supports used in earlier computations.

At $k = 114$, the reported support has 54 shells and begins with

$$12, 129, \dots,$$

while at $k = 120$ the reported support has 53 shells and begins with

$$12, 89, 150, \dots$$

The appearance of large internal gaps is therefore a notable numerical feature of the two computations. Its dependence on degree, candidate pool, solver tolerance, precision, and regularization remains to be studied.

The finite-field rank calculation provides independent algebraic information about the selected $k = 120$ support: its selected Fourier-evaluation columns are linearly independent, assuming the stated exact integer matrix and modular computation are correct.

Several important questions remain open:

- (1) Can the numerical positive weights be replaced by exact positive weights?
- (2) Is the observed support stable under increased arithmetic precision and changes in NNLS implementation?
- (3) Does the selected support generate a face of the full modular cone?
- (4) Is there a genuine asymptotic law governing the first exclusion gap?
- (5) What is the smallest support size capable of separating the relevant cumulative modular space?
- (6) Can an exact positive certificate be combined with a sufficiently large separating support to prove sign variation for an entire modular space?

These questions distinguish the experimentally observed sparse geometry from the theorems that follow from exact algebraic certificates.

16. CONCLUSION

Sparse non-negative optimization provides a computational framework for studying high-strength spherical cubatures supported on Leech-lattice shells.

The calculations reported here produce numerical candidates at strengths 115 and 121 with substantially non-contiguous supports. In both cases the supports exhibit pronounced internal exclusion gaps. For $k = 120$, an independent finite-field calculation verifies linear independence of the selected Fourier-evaluation columns, assuming the stated exact integer matrix and modular computation.

The modular-cone formulation also identifies precisely what would be needed for a rigorous sign-variation theorem: an exact strictly positive annihilating vector together with an appropriate injectivity or separation statement for the evaluation map. A floating-point residual alone is insufficient.

Likewise, the proposed quadratic horizon

$$m_*(k) \asymp \frac{k^2}{96}$$

is best viewed as a conjectural asymptotic scale suggested by a computational heuristic, rather than as an established law.

The main contribution is therefore a computationally testable framework for sparse modular cubature geometry, together with explicit statements of the additional certificates required to convert numerical observations into rigorous mathematical theorems.

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