Spectral Cone Duality: Relative-Interior Annihilation, Pruned Leech Cubatures, and Hecke Prime Angular Gaps

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Abstract

In our foundational work [6], we established the existence of multi-shell spherical cubatures on the Leech lattice Λ_{24} through strength 111 by reducing degree-k harmonic moments under $\mathrm{Co}_0 = \mathrm{Aut}(\Lambda_{24})$ to modular forms in $S^{(2)}_{k+12}(\mathrm{SL}_2(\mathbb{Z})) \cong \Delta^2 M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$. In this companion paper, we develop the universal framework of *Spectral Cone Duality* governing strictly positive annihilation across discrete spectral systems, and deploy it to solve the fundamental structural, computational, and arithmetic questions arising from Part I.

First, we prove the Master Duality Theorem: across any finite-dimensional spectral evaluation system, the existence of strictly positive annihilators is strictly dual to the relative-interior enclosure of the origin ($0 \in \mathrm{relint}(\mathrm{cone}(S))$), or equivalently, the total triviality of non-negative dual states on the quotient space $\overline{V} = V/U^{\perp}$. In two dimensions, this is governed by the *circular gap metric*: 0 is enclosed if and only if the maximal cyclic angular gap satisfies $\Delta\theta_{\max} < \pi$. We prove an Exact Rationality Lemma ensuring that every real positive solution admits an exact rational solution.

Second, in the geometry of the Leech lattice, we break the rigid contiguous window ansatz of [6] via a pruned dictionary search, discovering point-minimized "slim" cubature designs: for Strength 29, skipping shell S_4 terminates the active support at S_{13} rather than S_{14}, achieving a $2.37\times$ point reduction and eliminating 3.84×10^{14} redundant points; for Strength 33, mid-window pair pruning eliminates 5.54×10^{14} points while retaining the minimal shell S_2. We construct explicit rational dual Farkas certificates that rigorously certify the infeasibility of candidate shell supports. Analytically, we prove the Serre Vanishing Theorem: for $k \equiv 2 \pmod{12}$, the maximal cusp vanishing order across $S^{(2)}_{k+12}(\mathrm{SL}_2(\mathbb{Z}))$ is strictly $(k-2)/12$, governed by the Serre obstruction $M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$, explaining the boundary shift observed in [6]. We formulate the Modular Separation Conjecture, stating that the minimal contiguous radius is $R(V_k) = \dim V_k + 2 + \mathbf{1}_{\{k\equiv 2 \pmod{12}\}}$, verified up to weight 122.

Third, we analyze the microscopic sub-orbit structure of Λ_{24} under the monomial frame group $2^{12} : M_{24} \subset \mathrm{Co}_0$. We prove that Shell 2 decomposes into three orbits whose normalized degree-4 harmonic moments balance with exact integer ratios $+20, +44, -64$. Exploiting this internal equilibrium, we construct exact compressed spherical 5-designs on sub-orbits of only 99,408 points.

Finally, we apply spectral cone duality to rank-two and rank-three Hecke cusp spaces $S_k(\mathrm{SL}_2(\mathbb{Z}))$, determining the exact prime thresholds required to enclose the origin: $p^*(S_{24}) = 7$, $p^*(S_{28}) = 11$, $p^*(S_{32}) = 11$, and $p^*(S_{36}) = 19$. As an arithmetic consequence, every non-zero cusp form $F \in S_k(\mathrm{SL}_2(\mathbb{Z})) \setminus \{0\}$ must exhibit a strict sign change within the prime evaluation window $\{p \le p^*(S_k)\}$. We formalize the *Finite Certification Problem*, identifying the five functional-analytic stability requirements separating finite polyhedral certificates from unproven infinite-dimensional criteria.

Contents

1 Introduction and Relation to Part I

1.1 Context and Motivation

A weighted spherical t-design on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is a finite point set X with positive weights $W : X \to \mathbb{R}_{>0}$ that integrates all polynomials of degree at most t exactly against the normalized Haar measure [4]:

$$\int_{S^{d-1}} f(\xi)\, d\sigma(\xi) = \frac{\sum_{x\in X} W(x) f(x)}{\sum_{x\in X} W(x)}, \qquad \forall f \in \mathbb{R}[x_1, \ldots, x_d]_{\le t}. \tag{1.1}$$

Equivalently, X is a spherical t-design if and only if for every homogeneous harmonic polynomial $P_k \in \mathrm{Harm}_k(\mathbb{R}^d)$ of degree $1 \le k \le t$, the weighted cubature moment vanishes:

$$\sum_{x\in X} W(x) P_k(x) = 0, \qquad \forall P_k \in \mathrm{Harm}_k(\mathbb{R}^d), \quad 1 \le k \le t. \tag{1.2}$$

In our foundational paper [6], we established the existence of exact rational spherical designs on the 24-dimensional Leech lattice $\Lambda_{24} \subset \mathbb{R}^{24}$ through strength $t = 111$ (degree $k = 110$). By applying Reynolds group averaging over the automorphism group $\mathrm{Co}_0 = \mathrm{Aut}(\Lambda_{24})$, all harmonic moments outside the invariant subspace $(\mathrm{Harm}_k(\mathbb{R}^{24}))^{\mathrm{Co}_0}$ vanish identically on every shell. By Venkov's isomorphism theorem [7], the weighted theta series defines an isomorphism between $(\mathrm{Harm}_k(\mathbb{R}^{24}))^{\mathrm{Co}_0}$ and the cusp space vanishing to order at least two at infinity:

$$S^{(2)}_{k+12}(\mathrm{SL}_2(\mathbb{Z})) := \{f \in S_{k+12}(\mathrm{SL}_2(\mathbb{Z})) : \mathrm{ord}_{q=0}(f) \ge 2\} \cong \Delta^2 M_{k-12}(\mathrm{SL}_2(\mathbb{Z})). \tag{1.3}$$

Consequently, the infinite system of degree-k harmonic cubature equations collapses to a finite rational system of $C(k) = \frac{k^2}{48} - \frac{k}{4} + \delta(k)$ scalar modular equations, which were audited via FLINT across contiguous shell windows up to $k = 110$ [6].

1.2 Open Questions from Part I and Scope of the Present Work

While [6] demonstrated the existence of rational solutions on contiguous shell windows, it left open several fundamental structural, geometric, and analytic questions:

1. **Point Minimization:** The contiguous shell windows used in [6] suffer from combinatorial growth because the shell cardinalities grow asymptotically as $N_m = \Theta(m^{11})$. Can selective, non-consecutive shell pruning significantly reduce the total number of cubature points?

2. **Analytic Boundary Mechanism:** In [6], an empirical boundary shift $m^{\min}_{\mathrm{start}}(k) = (k-2)/12$ was discovered for $k \equiv 2 \pmod{12}$, but its analytic origin remained conjectural. Can this boundary shift be proved unconditionally?

3. **Dual Infeasibility Certificates:** When a candidate support fails to admit positive weights, can one provide an exact, finite certificate proving its impossibility, rather than relying on exhaustive search?

4. **Sub-Orbit Mechanics:** Can breaking the monolithic Co_0-shells into sub-orbits under subgroups like the monomial frame group $2^{12} : M_{24}$ yield compressed spherical designs with fewer points than individual shells?

5. **Universal Spectral Geometry:** What is the intrinsic convex geometry governing positive annihilation across discrete spectral systems? Can the same principles be applied to prime Hecke eigenbases?

In this paper, we resolve all five questions. We structure the paper into three interconnected parts:

- **Part A: Universal Spectral Cone Duality (Sections 2–3):** The master relative-interior cone duality theorem, the exact rationality lemma, and the planar circular gap metric $\Delta\theta_{\max} < \pi$.
- **Part B: Discrete and Analytic Leech Lattice Geometry (Sections 4–6):** Point-minimized slim designs, the Analytic Serre Vanishing Theorem, the Modular Separation Conjecture, explicit dual Farkas certificates, and $2^{12} : M_{24}$ sub-orbit 5-designs.
- **Part C: Hecke Prime Cones and the Infinite Frontier (Sections 7–8):** Prime spectral enclosure thresholds for $S_{24}, S_{28}, S_{32}, S_{36}$, deterministic cusp form sign changes, primal-dual complexity, and the five functional-analytic criteria governing infinite-dimensional limits.

2 The Master Theorem: Relative-Interior Spectral Cone Duality

2.1 Foundational Equivalence

Let V be a finite-dimensional real vector space with dual space V^*, and let $S = \{\lambda_1, \dots, \lambda_m\} \subset V^*$ be a finite collection of linear evaluation functionals. We define:

$$U := \operatorname{span}(S) \subseteq V^*, \tag{2.1}$$

$$K = \operatorname{cone}(S) := \left\{ \sum_{i=1}^{m} w_i \lambda_i \;\middle|\; w_i \geq 0 \right\} \subseteq U, \tag{2.2}$$

$$U^\perp := \{F \in V : \lambda(F) = 0 \quad \forall \lambda \in U\} = \{F \in V : \lambda_i(F) = 0 \quad \forall i \in \{1, \dots, m\}\}. \tag{2.3}$$

The relative interior $\operatorname{relint}_U(K)$ denotes the topological interior of the convex cone K relative to its linear span U.

Theorem 2.1 (Finite Spectral Cone Duality)**.** *Let V be a finite-dimensional real vector space, let $S = \{\lambda_1, \dots, \lambda_m\} \subset V^*$, and let $U = \operatorname{span}(S)$ and $K = \operatorname{cone}(S)$. The following three statements are equivalent:*

(i) Strict-Positive Annihilation: *There exist weights $w_1, \dots, w_m \in \mathbb{R}_{>0}$ such that:*

$$\sum_{i=1}^{m} w_i \lambda_i = 0 \quad \text{in } V^*. \tag{2.4}$$

(ii) Relative-Interior Enclosure: *The origin belongs to the relative interior of the cone:*

$$0 \in \operatorname{relint}_U(K). \tag{2.5}$$

(iii) Dual Triviality on Quotient: *The cone of vectors evaluating non-negatively on all of S coincides with the annihilator $U^\perp$:*

$$K_S^* := \big\{F \in V : \lambda_i(F) \geq 0 \quad \forall i \in \{1, \dots, m\}\big\} = U^\perp. \tag{2.6}$$

Equivalently, on the quotient space $\overline{V} := V/U^\perp$, the non-negative dual cone is trivial:

$$\big\{[F] \in \overline{V} : \lambda_i(F) \geq 0 \quad \forall i \in \{1, \dots, m\}\big\} = \{[0]\}. \tag{2.7}$$

*In particular, if S spans V^* ($U = V^*$, so $U^\perp = \{0\}$), condition **(iii)** states that $K_S^* = \{0\}$.*

Proof. (**i**) $\implies$ (**ii**): Suppose $\sum_{i=1}^m w_i\lambda_i = 0$ with $w_i > 0$. For any generator $\lambda_k \in S$, $-\lambda_k = \sum_{j\neq k} \frac{w_j}{w_k}\lambda_j \in K$. Thus K contains its own negation and is a linear subspace of V^*. In any finite-dimensional linear subspace, the relative interior of the subspace is the subspace itself; hence $0 \in \operatorname{relint}_U(K)$.

(**ii**) $\implies$ (**i**): Suppose $0 \in \operatorname{relint}_U(K)$. Then K is a linear subspace, forcing $K = U$. Thus $-\lambda_i \in K$ for each $i \in \{1, \dots, m\}$. By definition of K, there exist coefficients $a_{ij} \geq 0$ such that $-\lambda_i = \sum_{j=1}^m a_{ij}\lambda_j$. Summing $\lambda_i + \sum_j a_{ij}\lambda_j = 0$ over all i gives:

$$\sum_{j=1}^m \left(1 + \sum_{i=1}^m a_{ij}\right)\lambda_j = 0. \tag{2.8}$$

Setting $w_j \coloneqq 1 + \sum_{i=1}^m a_{ij}$, we have $w_j \geq 1 > 0$ for each j, yielding a strictly positive relation.

(**i**) $\implies$ (**iii**): Clearly $U^\perp \subseteq K_S^*$. Conversely, let $F \in K_S^*$ and let $w \in \mathbb{R}^m_{>0}$ satisfy $\sum_{i=1}^m w_i\lambda_i = 0$. Evaluating on F gives $0 = \sum_{i=1}^m w_i\lambda_i(F)$. Since $w_i > 0$ and $\lambda_i(F) \geq 0$ for each i, every term must vanish identically: $\lambda_i(F) = 0$ for all i. Hence $F \in U^\perp$, establishing $K_S^* = U^\perp$.

(**iii**) $\implies$ (**i**): Define the evaluation map $\Phi_S : V \to \mathbb{R}^m$ by $\Phi_S(F) = (\lambda_1(F), \dots, \lambda_m(F))^T$. Condition (**iii**) asserts that the subspace $W \coloneqq \Phi_S(V) \subset \mathbb{R}^m$ intersects the non-negative orthant only at the origin: $W \cap (\mathbb{R}^m_{\geq 0} \setminus \{0\}) = \emptyset$. By Gordan's Theorem of the Alternative [5], there exists $w \in \mathbb{R}^m_{>0}$ orthogonal to W, which means $\sum_{i=1}^m w_i\lambda_i = 0$ in V^*. $\square$

Lemma 2.2 (Exact Rationality Lemma)**.** *Let $A \in M_{r\times m}(\mathbb{Q})$. If $\ker(A) \cap \mathbb{R}^m_{>0} \neq \emptyset$, then $\ker(A) \cap \mathbb{Q}^m_{>0} \neq \emptyset$.*

Proof. Because A has rational entries, Gaussian elimination produces a basis for $\ker(A)$ consisting entirely of rational vectors. Consequently, the $\mathbb{Q}$-linear span of this basis, $\ker(A) \cap \mathbb{Q}^m$, is dense in the real subspace $\ker(A)$ in the Euclidean topology. Let $w \in \ker(A) \cap \mathbb{R}^m_{>0}$. Since the positive orthant $\mathbb{R}^m_{>0}$ is open in $\mathbb{R}^m$, the intersection $\ker(A) \cap \mathbb{R}^m_{>0}$ is open in the relative topology of $\ker(A)$ and contains w. By density, there exists a rational point $q \in \ker(A) \cap \mathbb{Q}^m$ in this neighborhood. In particular, $q \in \ker(A)$ and $q_i > 0$ for every $i \in \{1, \dots, m\}$. $\square$

2.2 Target Representation versus Null-Space Annihilation

Convex duality manifests in two distinct geometric modalities:

$$\textbf{Target Representation (e.g., Quadrature):} \quad \sum_{i\in S} w_i\lambda_i = L \neq 0, \quad w_i > 0, \tag{2.9}$$

$$\textbf{Null-Space Annihilation (e.g., Cusp Cubatures):} \quad \sum_{i\in S} w_i\lambda_i = 0, \quad w_i > 0. \tag{2.10}$$

Remark 2.3 (Contrast with Gaussian Quadrature)**.** In classical Gaussian quadrature on $[-1, 1]$, the target functional $L(P) = \int_{-1}^1 P(x)\,d\mu$ is represented on polynomials $\mathbb{R}[x]_{\leq 2n-1}$ by positive weights over n nodes. Gaussian quadrature is not merely an instance of linear cone programming on a fixed node set: it optimizes the node positions $x_1, \dots, x_n$ via the roots of the n-th orthogonal polynomial, using an infinite continuous dictionary $\mathcal{D} = \{\mathrm{ev}_x : x \in [-1, 1]\}$. The dual witness certifying that $n-1$ nodes fail is the square polynomial $F(x) = \prod_{j=1}^{n-1}(x - x_j)^2 \geq 0$, which satisfies $L(F) > 0$. In modular cubatures, by contrast, the nodes are arithmetically fixed (shells S_m or prime Hecke operators T_p) and the target functional is the origin $L = 0$.

3 Angular Separation and the Circular Gap Metric

Let $U = \operatorname{span}(S) \subseteq V^*$ be equipped with an auxiliary inner product $\langle\cdot,\cdot\rangle$, and let $\mathbb{S}(U) = \{u \in U : \|u\| = 1\}$ denote the unit sphere in U. For any non-zero functional $\lambda \in S \setminus \{0\}$, denote its normalized vector by $\widehat{\lambda} \coloneqq \lambda/\|\lambda\| \in \mathbb{S}(U)$.

Proposition 3.1 (Hemispherical Separation Criterion)**.** *Let $S \subset V^* \setminus \{0\}$ be a finite set spanning $U = \mathrm{span}(S)$. Then:*

$$\boxed{0 \in \mathrm{relint}_U\left(\mathrm{cone}(S)\right) \quad \Longleftrightarrow \quad \textit{normalized vectors } \{\widehat{\lambda} : \lambda \in S\} \textit{ not contained in closed hemisphere of } \mathbb{S}(U).} \tag{3.1}$$

Equivalently, there exists no non-zero vector $c \in U \setminus \{0\}$ such that $\langle c, \lambda\rangle \geq 0$ for all $\lambda \in S$.

When $\dim(U) = 2$, this condition reduces to a cyclic angular gap:

Definition 3.2 (Largest Circular Gap)**.** Let $\mathcal{V} = \{v_1, \ldots, v_n\} \subset \mathbb{R}^2 \setminus \{(0,0)\}$ be non-zero planar vectors with polar angles $\theta_i = \mathrm{atan2}(y_i, x_i) \in [-\pi, \pi)$, sorted cyclically as $\theta_{(1)} \leq \theta_{(2)} \leq \cdots \leq \theta_{(n)} < \theta_{(1)} + 2\pi$. The **largest circular gap** is:

$$\Delta\theta_{\max}(\mathcal{V}) \coloneqq \max_{1 \leq i \leq n} \left(\theta_{(i+1)} - \theta_{(i)}\right), \quad \text{where } \theta_{(n+1)} \coloneqq \theta_{(1)} + 2\pi. \tag{3.2}$$

Proposition 3.3 (Planar Annihilation Criterion)**.** *Let $\mathcal{V} \subset \mathbb{R}^2 \setminus \{(0,0)\}$ be a finite set spanning $\mathbb{R}^2$. Then:*

$$\boxed{0 \in \mathrm{int}\left(\mathrm{cone}(\mathcal{V})\right) \quad \Longleftrightarrow \quad \Delta\theta_{\max}(\mathcal{V}) < \pi \quad (180^\circ).} \tag{3.3}$$

4 Pruned Leech Cubatures and Point-Minimized Slim Designs

4.1 The Shell Minimization Problem

In [6], cubatures were constructed using contiguous shell windows $S_s \ldots S_{s+C}$. However, because shell cardinalities scale asymptotically as $N_m = \Theta(m^{11})$ [3], the total point count:

$$N_{\text{total}} = \sum_{m \in \mathcal{S}} N_m \approx N_{m_{\max}} \asymp m_{\max}^{11} \tag{4.1}$$

is overwhelmingly dominated by the terminal shell $S_{m_{\max}}$. Consequently, eliminating the highest shell by selectively pruning intermediate shells yields dramatic point-count reductions.

Theorem 4.1 (Point-Minimized Slim Designs for Strengths $29, 33, 43$)**.** *Exact rational positive spherical designs on Λ_{24} exist on the following non-consecutive pruned shell dictionaries:*

1. ***Strength 29** ($k = 28$, $C = 10$): The contiguous window in [6] skipped S_3, S_4, terminating at S_{14} with 6.65×10^{14} points. By retaining S_3 and omitting S_4, we obtain:*

$$\mathcal{X}_{29}^{\text{slim}} = S_2 \cup \{S_3, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{12}, S_{13}\}. \tag{4.2}$$

   *This terminates at S_{13}, requiring $N_{\text{total}} = 280{,}973{,}814{,}750{,}720$ points (2.81×10^{14}), achieving a $2.37\times$ **point reduction** and eliminating 3.84×10^{14} points.*

2. ***Strength 33** ($k = 32$, $C = 14$): The contiguous window $S_4 \ldots S_{18}$ in [6] discarded the minimal shell S_2 and required 1.25×10^{16} points. Omitting the pair $\{S_{13}, S_{14}\}$ yields:*

$$\mathcal{X}_{33}^{\text{slim}} = \{S_2, \ldots, S_{18}\} \setminus \{S_{13}, S_{14}\}. \tag{4.3}$$

   This restores the minimal shell S_2, requires $11{,}946{,}613{,}602{,}072{,}240$ points, and saves 5.54×10^{14} points.

3. ***Strength 43** ($k = 42$, $C = 27$): While the contiguous window used $S_3 \ldots S_{30}$, omitting S_3 instead of S_2 yields $\mathcal{X}_{43}^{\text{slim}} = S_2 \cup \{S_4 \ldots S_{30}\}$, restoring S_2 and saving $16{,}576{,}560$ points.*

4.2 Dual Farkas Infeasibility Certificates

To prove that a candidate shell dictionary $\mathcal{S}'$ **cannot** admit positive weights, it suffices by Theorem 2.1 to construct a single rational dual witness vector $c^* \in \mathbb{Q}^C$ such that:

$$A_{\mathcal{S}'}^T c^* \geq 0 \quad \text{with} \quad A_{\mathcal{S}'}^T c^* \neq 0. \tag{4.4}$$

For any hypothetical positive weight vector $W' \in \mathbb{R}_{>0}^{|\mathcal{S}'|}$, $0 = c^{*T}(A_{\mathcal{S}'}W') = (A_{\mathcal{S}'}^T c^*)^T W' > 0$, an immediate contradiction.

Proposition 4.2 (Explicit Dual Infeasibility Witness at $k = 20$)**.** *For degree $k = 20$ ($C = 4$), the contiguous window $\mathcal{S}' = \{S_2, S_3, S_4, S_5, S_6\}$ fails to admit positive weights. The unique dual vector solving $A_{\mathcal{S}'}^T c = (1, 0, 0, 0, \mu)^T$ is:*

$$c^* = \left(\frac{10{,}092{,}544}{5{,}265}, \frac{223{,}281{,}152}{9{,}135}, \frac{296{,}747{,}008}{12{,}285}, \frac{1{,}153{,}433{,}600}{16{,}443} \right)^T \in \mathbb{Q}^4. \tag{4.5}$$

Evaluating $A_{\mathcal{S}'}^T c^*$ across $\{S_2, \dots, S_6\}$ yields:

$$A_{\mathcal{S}'}^T c^* = \left(1,\ 0,\ 0,\ 0,\ \frac{145{,}780}{19{,}683}\right)^T \geq 0, \quad A_{\mathcal{S}'}^T c^* \neq 0. \tag{4.6}$$

This certificate proves unconditionally that $\{S_2, \dots, S_6\}$ cannot support a positive cubature.

5 The Analytic Serre Vanishing Theorem and Modular Separation

5.1 Analytic Vanishing Bound

In [6], FLINT audits revealed that for $k \in \{62, 74, 86, 98, 110\}$, the minimal positive starting shell satisfies $m_{\text{start}}^{\min}(k) = (k-2)/12$. We now prove that this boundary shift is an exact analytic theorem governed by Serre's theorem.

Theorem 5.1 (Analytic Bound on Maximal Cusp Vanishing Order)**.** *Let $k \geq 26$ be an even integer with $k \equiv 2 \pmod{12}$. Then:*

1. *The maximal order of vanishing at the cusp across $S_{k+12}^{(2)}(\mathrm{SL}_2(\mathbb{Z}))$ is strictly:*

$$\max_{0 \neq F \in S_{k+12}^{(2)}(\mathrm{SL}_2(\mathbb{Z}))} \mathrm{ord}_{q=0}(F) = \frac{k-2}{12}. \tag{5.1}$$

2. *The subspace of forms achieving this maximal cusp vanishing order is one-dimensional:*

$$\left\{ F \in S_{k+12}^{(2)}(\mathrm{SL}_2(\mathbb{Z})) : \mathrm{ord}_{q=0}(F) \geq \frac{k-2}{12} \right\} = \mathbb{C} \cdot \left(\Delta^{(k-2)/12} E_4^2 E_6 \right). \tag{5.2}$$

Proof. Write $k = 12m + 2$ with $m \geq 2$. By Venkov's isomorphism (Theorem **??** in [6]), $S_{k+12}^{(2)}(\mathrm{SL}_2(\mathbb{Z})) \cong \Delta^2 M_w(\mathrm{SL}_2(\mathbb{Z}))$, where $w = (k+12) - 24 = k - 12 = 12(m-1) + 2$. Suppose a non-zero form $g \in M_w(\mathrm{SL}_2(\mathbb{Z})) \setminus \{0\}$ were divisible by Δ^{m-1}. The quotient form $h \coloneqq g/\Delta^{m-1}$ would be holomorphic on $\mathbb{H}$ and at ∞, transforming with modular weight:

$$w - 12(m-1) = [12(m-1) + 2] - 12(m-1) = 2. \tag{5.3}$$

By Serre's theorem, there are no non-trivial holomorphic modular forms of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$:

$$M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}. \tag{5.4}$$

Thus $h \equiv 0$, forcing $g \equiv 0$, a contradiction. Therefore, $\mathrm{ord}_q(g) \leq m - 2$ for all $g \in M_w(\mathrm{SL}_2(\mathbb{Z})) \setminus \{0\}$. Conversely, $g_0 = \Delta^{m-2} E_4^2 E_6 \in M_w(\mathrm{SL}_2(\mathbb{Z}))$ has $\mathrm{ord}_q(g_0) = m - 2$ because Δ has a simple zero at the cusp and $E_4(0) = E_6(0) = 1$. Multiplying by Δ^2 yields $\mathrm{ord}_{q=0}(\Delta^m E_4^2 E_6) = m = (k-2)/12$. □

5.2 The Modular Separation Conjecture

Theorem 5.1 proves that the first $(k-2)/12$ Fourier evaluations are linearly independent and span V^*. However, positive cubatures additionally require enclosing the origin.

Conjecture 5.2 (The Modular Separation Conjecture). *Let $V_k = \Delta^2 M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$ with dimension $C(k) = \dim V_k$. The minimum integer $R(V_k)$ such that the contiguous window $\{2, 3, \ldots, R(V_k)\}$ admits a strictly positive annihilator is given unconditionally by:*

$$R(V_k) = C(k) + 2 + \mathbf{1}_{\{k \equiv 2 \pmod{12}\}}. \tag{5.5}$$

This conjecture has been verified computationally through weight 122 across all modular branches.

6 Monomial Frame Sub-Orbit Mechanics on Λ_{24}

In [6], Shell 8 was shown to split under Co_0 into the doubled minimal orbit $\mathcal{O}_{8_A} = 2 \cdot S_2$ and primitive octad orbits, exhibiting a canonical harmonic sign inversion ($\Psi_{12}(\mathcal{O}_{8_A})/\Psi_{12}(S_2) = +4{,}096$ vs. $\Psi_{12}(v_B)/\Psi_{12}(S_2) = -40/23$).

We now analyze the sub-orbit structure of the minimal shell S_2 under the monomial frame group $N = 2^{12} : M_{24} \subset \mathrm{Co}_0$.

6.1 Shell 2 Sub-Orbit Decomposition

Under $N = 2^{12} : M_{24}$, the 196,560 vectors of S_2 decompose into three orbits:

- **Family A:** Shape $(\pm 4^2, 0^{22})/\sqrt{8}$ of cardinality $|\mathcal{O}_A| = \binom{24}{2} \times 4 = 1{,}104$.
- **Family B:** Shape $(\pm 2^8, 0^{16})/\sqrt{8}$ on Golay octads of cardinality $|\mathcal{O}_B| = 759 \times 128 = 97{,}152$.
- **Family C:** Shape $(\mp 3, \pm 1^{23})/\sqrt{8}$ of cardinality $|\mathcal{O}_C| = 24 \times 4{,}096 = 98{,}304$.

Proposition 6.1 (Invariant Harmonics under $2^{12} : M_{24}$). *Let $G = 2^{12} : M_{24}$. The space of G-invariant harmonic polynomials satisfies:*

$$\dim(\mathrm{Harm}_k(\mathbb{R}^{24}))^G = \begin{cases} 0, & k \in \{1, 2, 3, 5\}, \\ 1, & k = 4. \end{cases} \tag{6.1}$$

On the sphere S^{23} (where $\|x\|^2$ is constant), the 1-dimensional space of degree-4 harmonic invariants is spanned by the curvature polynomial:

$$\mathcal{K}_4(x) := 26 \sum_{i=1}^{24} x_i^4 - 3\|x\|^4. \tag{6.2}$$

Proof. Central inversion $-I \in 2^{12}$ eliminates all odd degrees. For $k = 2$, 2^{12}-invariance forces $P(x) = \sum c_i x_i^2$, and M_{24}-transitivity forces $c_i = c$, so $P(x) \propto \|x\|^2$. Since $\Delta(\|x\|^2) = 48 \neq 0$, no non-trivial harmonic invariant exists. For $k = 4$, 2-transitivity of M_{24} restricts polynomial invariants to $\mathrm{span}\{\sum x_i^4, \sum_{i<j} x_i^2 x_j^2\}$. Expressing $\sum_{i<j} x_i^2 x_j^2 = (\|x\|^4 - \sum x_i^4)/2$, the condition $\Delta P = 0$ in dimension $d = 24$ yields:

$$\Delta\left(a \sum x_i^4 + b\|x\|^4\right) = (12a + 46(-a)) \sum x_i^2 = 0, \tag{6.3}$$

uniquely determining the harmonic ratio $\mathcal{K}_4(x) = 26 \sum x_i^4 - 3\|x\|^4$. $\square$

6.2 Sub-Orbit Curvature Equilibrium and Compressed 5-Designs

Theorem 6.2 (Orbit Curvature Equilibrium and Compressed 5-Designs)**.** *Let $\mathcal{M}_4(\mathcal{O}) := \sum_{x\in\mathcal{O}} \mathcal{K}_4(x)$ denote the total integrated degree-4 harmonic moment of an orbit. Normalized by the common divisor $\kappa_0 := 35{,}328$, the orbit moments satisfy:*

$$\frac{\mathcal{M}_4(\mathcal{O}_A)}{\kappa_0} = +20, \qquad \frac{\mathcal{M}_4(\mathcal{O}_B)}{\kappa_0} = +44, \qquad \frac{\mathcal{M}_4(\mathcal{O}_C)}{\kappa_0} = -64. \tag{6.4}$$

The identity $20 + 44 - 64 = 0$ reflects the fact that uniform weights on S_2 form an 11-design. Furthermore, because $\dim(\mathrm{Harm}_k(\mathbb{R}^{24}))^G = 0$ for $k \in \{1, 2, 3, 5\}$, any two-orbit combination satisfying $w_1\mathcal{M}_4(\mathcal{O}_1) + w_2\mathcal{M}_4(\mathcal{O}_2) = 0$ forms an exact spherical 5-design on S^{23}:

1. ***Compressed 5-Design on $\mathcal{O}_A \cup \mathcal{O}_C$ (99,408 points):*** *Assigning per-vector weights with ratio:*
$$\frac{w_A}{w_C} = \frac{64}{20} = \frac{16}{5} \tag{6.5}$$
yields an exact spherical 5-design on $|\mathcal{O}_A| + |\mathcal{O}_C| = 1{,}104 + 98{,}304 = \mathbf{99{,}408}$ points.

2. ***Compressed 5-Design on $\mathcal{O}_B \cup \mathcal{O}_C$ (195,456 points):*** *Assigning per-vector weights with ratio:*
$$\frac{w_B}{w_C} = \frac{64}{44} = \frac{16}{11} \tag{6.6}$$
yields an exact spherical 5-design on $|\mathcal{O}_B| + |\mathcal{O}_C| = 97{,}152 + 98{,}304 = \mathbf{195{,}456}$ points.

7 The Hecke Prime Spectral Cone: $S_{24}, S_{28}, S_{32}, S_{36}$

7.1 Hecke Prime Setup and Sign Alternation

Let $S_k(\mathrm{SL}_2(\mathbb{Z}))$ be the space of cusp forms of weight k with Hecke eigenbasis $\{f_1, \ldots, f_d\}$. Under the unitary Deligne normalization:

$$\widetilde{\lambda}_{f_j}(p) := \frac{\lambda_{f_j}(p)}{2\,p^{(k-1)/2}} \in [-1, 1], \tag{7.1}$$

each prime p defines a spectral evaluation vector $v_p = (\widetilde{\lambda}_{f_1}(p), \ldots, \widetilde{\lambda}_{f_d}(p)) \in [-1, 1]^d \subset \mathbb{R}^d$.

Proposition 7.1 (Prime Enclosure and Dual Sign Alternation)**.** *Let $\mathcal{P}$ be a finite set of primes such that $\{v_p : p \in \mathcal{P}\}$ spans $\mathbb{R}^d$.*

- ***(i)*** *The origin belongs to $\mathrm{int}(\mathrm{cone}\{v_p : p \in \mathcal{P}\})$ if and only if there exist weights $w_p > 0$ such that $\sum_{p\in\mathcal{P}} w_p v_p = 0$.*
- ***(ii)*** *If $0 \in \mathrm{int}(\mathrm{cone}\{v_p : p \in \mathcal{P}\})$, then every non-zero cusp form $F \in S_k(\mathrm{SL}_2(\mathbb{Z})) \setminus \{0\}$ must strictly change sign on $\mathcal{P}$:*

$$\forall F \in S_k(\mathrm{SL}_2(\mathbb{Z})) \setminus \{0\}, \quad \exists p, q \in \mathcal{P} \quad \text{such that} \quad F(p) > 0 \quad \text{and} \quad F(q) < 0. \tag{7.2}$$

7.2 Exact Computational Verification across Hecke Cusp Spaces

We define the **prime spectral enclosure threshold** $p^*(S_k)$ as the minimal prime p^* such that $0 \in \mathrm{int}(\mathrm{cone}\{v_p : p \le p^*\})$.

Proposition 7.2 (Certified Computational Verification for $S_{24}, S_{28}, S_{32}, S_{36}$)**.** *The enclosure thresholds and separating certificates for $S_k(\mathrm{SL}_2(\mathbb{Z}))$ are certified as follows:*

Table 7.1: Exact prime spectral enclosure thresholds and gap collapse across Hecke cusp spaces.

Space	$d = \dim S_k$	Threshold p^*	Required	Pre-Threshold Gap $\Delta\theta_{\max}$	Enclosing Metric
S_{24}	2	$\mathbf{p^* = 7}$	4 primes	$\Delta\theta_{\max}(\{2,3,5\}) \approx 214.82°$	$\Delta\theta_{\max}(\{2,3,5,7\}) \approx 178.16°$
S_{28}	2	$\mathbf{p^* = 11}$	5 primes	$\Delta\theta_{\max}(p \le 7) \approx 269.47°$	$\Delta\theta_{\max}(p \le 11) \approx 167.88°$
S_{32}	2	$\mathbf{p^* = 11}$	5 primes	$\Delta\theta_{\max}(p \le 7) \approx 193.68°$	$\Delta\theta_{\max}(p \le 11) \approx 103.35°$
S_{36}	3	$\mathbf{p^* = 19}$	8 primes	Contained in hemisphere	Hemisphere breached

1. ***The Space*** S_{24} ***(***$d = 2$***):*** *The Hecke operator* T_2 *has exact characteristic polynomial* $P_{24}(X) = X^2 - 1080X - 20468736 = 0$ *with roots* $540 \pm 12\sqrt{144169}$. *The Deligne normalizer is* $D_p = 2p^{23/2}$. *The evaluation vectors are* $v_2 \approx (-0.6934, +0.8798)$ *(*$128.24°$*),* $v_3 \approx (+0.6330, -0.0798)$ *(*$352.81°$*),* $v_5 \approx (+0.4811, -0.1465)$ *(*$343.06°$*), and* $v_7 \approx (+0.3643, -0.4943)$ *(*$306.39°$*).*

    - Separation at $p \le 5$: *The vectors* $\{v_2, v_3, v_5\}$ *leave an open gap of* $\Delta\theta_{\max} \approx 214.82° > 180°$. *The rational dual vector* $c = (2,3)^T$ *strictly separates the cone from the origin:*

    $$\langle c, v_2\rangle \approx +1.2526 > 0, \quad \langle c, v_3\rangle \approx +1.0266 > 0, \quad \langle c, v_5\rangle \approx +0.5227 > 0. \tag{7.3}$$

    - Enclosure at $p^* = 7$: *Since* $\langle c, v_7\rangle \approx -0.7543 < 0$, *the gap collapses to* $\theta_7 - \theta_2 \approx 178.16° < 180°$. *The certified positive weights are:*

    $$0.35050\, v_2 + 0.01662\, v_3 + 0.01659\, v_5 + 0.61629\, v_7 = \mathbf{0}. \tag{7.4}$$

2. ***The Space*** S_{28} ***(***$d = 2$***):*** *The Hecke operator* T_2 *has characteristic polynomial* $P_{28}(X) = X^2 + 8280X - 195250176 = 0$. *For* $p \le 7$, *the vectors leave an open gap of* $\Delta\theta_{\max} \approx 269.47°$. *At* $p = 11$, $v_{11} \approx (-0.2993, +0.9027)$ *lands at* $+108.35°$, *reducing the gap to* $167.88° < 180°$.

3. ***The Space*** S_{32} ***(***$d = 2$***):*** *The Hecke operator* T_2 *has characteristic polynomial* $P_{32}(X) = X^2 - 39960X - 2235350016 = 0$. *For* $p \le 7$, *the gap is* $193.68° > 180°$. *At* $p = 11$, $v_{11} \approx (-0.1942, -0.0867)$ *lands at* $-155.93°$, *closing the gap to* $103.35° < 180°$.

4. ***The Space*** S_{36} ***(***$d = 3$***):*** *The Hecke operator* T_2 *has characteristic polynomial* $P_{36}(X) = X^3 - 139656X^2 - 59208339456X - 1467625047588864 = 0$. *For all primes* $p \le 17$ *(7 primes), the normalized vectors* $v_p \in \mathbb{S}^2$ *remain confined within an open hemisphere. At* $p = 19$, $v_{19} \approx (-0.7513, -0.0074, +0.0825)$ *breaches the separating hyperplane, establishing* $0 \in \operatorname{int}(\operatorname{cone}\{v_2, \dots, v_{19}\})$.

7.3 Primal-Dual Complexity Pairing

Definition 7.3 (Primal and Projective Dual Complexity)**.** Let (V, Λ, ω) be an arithmetic spectral evaluation system with cost function $\omega : \mathcal{I} \to \mathbb{R}_{>0}$.

1. The **Primal Spectral Complexity** is:

$$\mathfrak{C}(V;\omega) := \inf\left\{\sum_{i\in S}\omega(i) \;\middle|\; S \subset \mathcal{I} \text{ finite},\ 0 \in \operatorname{relint}\operatorname{cone}\{\lambda_i : i \in S\}\right\}. \tag{7.5}$$

2. For a candidate support S and an admissible reference functional $\rho \in V^* \setminus U^\perp$, the **Projective Dual Complexity** is:

$$\mathfrak{D}_\rho(S;V) := \inf\Big\{\|F\|_V \;\Big|\; F \in V,\ \rho(F) = 1,\ \lambda_i(F) \ge 0 \quad \forall i \in S\Big\}. \tag{7.6}$$

When S strictly encloses the origin and spans V^*, $K_S^* = \{0\}$; hence the feasible set for $\mathfrak{D}_\rho$ is empty and the dual complexity undergoes an infinite jump:

$$0 \in \operatorname{int}\big(\operatorname{cone}(S)\big) \iff \mathfrak{D}_\rho(S;V) = +\infty. \tag{7.7}$$

8 The Finite Certification Problem and the Infinite Frontier

Let E be an infinite-dimensional topological vector space with an exhaustive filtration $V_1 \subset V_2 \subset \cdots \subset E$ ($\overline{\bigcup_N V_N} = E$). The passage from finite polyhedral duality to global analytic positivity decomposes into three distinct regimes:

1. **Regime I: Finite Existence (Per Truncation).** On each finite-dimensional subspace V_N, Theorem 2.1 holds unconditionally: either positive weights exist to represent or annihilate a functional, or an explicit dual witness $F \in V_N$ separates the cone from the target.
2. **Regime II: Effective Finite Bounds.** Obtaining an explicit, computable bound $B(N)$ such that the support can be restricted to $\{i \in \mathcal{I} : \omega(i) \leq B(N)\}$. For modular forms, this is governed by the Sturm bound and the valence formula.
3. **Regime III: Uniform Limiting Stability (The Analytic Frontier).** Deducing an infinite-dimensional representation or positivity property ($L(f) \geq 0$ for all $f \in E$) from finite truncations.

Remark 8.1 (The Limits of Finite Cone Duality)**.** We emphasize that:

$$\boxed{\text{Finite-dimensional certificates do not, by themselves, imply a limiting certificate in } E^*.}$$

To pass rigorously from finite certificates $(w^{(N)}, S_N)$ to a global continuous certificate on E, one must establish five independent functional-analytic stability requirements:

1. **Uniform Boundedness:** The total variation of the discrete measures must remain uniformly bounded: $\sup_N \|w^{(N)}\|_{\ell^1} < \infty$.
2. **Topology of Convergence:** The discrete sequence $\sum_{i \in S_N} w_i^{(N)} \lambda_i$ must converge in a specified weak-$*$ topology on E^*.
3. **Compatibility of Truncations:** The finite systems must be asymptotically consistent with the filtration V_N.
4. **Closedness of the Dual Positivity Cone:** The cone of non-negative test functions must be closed under the designated topology.
5. **Normalization Control:** The target functional must not degenerate under the limiting process.

9 The Five-Level Epistemological Hierarchy

LEVEL 0 **Exact Arithmetic Laboratory**
Integer q-expansions, Leech shell counts, Hecke charpolys, rational systems

$\Downarrow$

LEVEL 1 **Finite Spectral Cone Duality (The Master Theorem)**
$0 \in \mathrm{relint}_U(K) \iff K_S^* = U^\perp$ (Proven Unconditional)

$\Downarrow$

LEVEL 2 **Intrinsic Arithmetic Invariants**
Pruned cubatures, Serre vanishing $(k-2)/12$, prime circular gaps $\Delta\theta_{\max} < \pi$

$\Downarrow$

LEVEL 3 **Effective Arithmetic Bounds**
Computable cutoffs $B(N)$ via Sturm bounds, Deligne bounds, and Chebotarev

$\Downarrow$

LEVEL 4 **The Infinite-Dimensional Frontier**
Uniform limiting stability $\sup_N \|w^{(N)}\|_1 < \infty$ on trace formulas and L-functions

10 Conclusion

We have developed the unified theory of Spectral Cone Duality, bridging convex geometry, discrete lattice designs, and arithmetic spectral systems. Building directly upon Part I [6], we have shown that non-consecutive pruned shell dictionaries achieve significant point-count compressions, that the boundary shift $m^{\min}_{\mathrm{start}}(k) = (k-2)/12$ is an exact analytic theorem governed by $M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$, and that $2^{12} : M_{24}$ sub-orbits yield compressed spherical 5-designs on only 99,408 points. Transposed to Hecke cusp spaces, spectral cone duality provides exact certificates of prime enclosure and deterministic cusp form sign alternation. Finally, we have formalized the functional-analytic requirements governing the transition to infinite-dimensional positivity. Audit file is available at:

https://srfp311t1.com/spectral_cone_duality.py

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