# A Detailed Reconstruction of the Periodic Uniqueness Argument
# for the Leech Lattice in Dimension 24

An Expository Analysis of the Equality and Structure-Factor Argument of Cohn–Kumar–Miller–Radchenko–Viazovska

SRFP311T1 Collaboration

September 2026

**Abstract**

In their 2017 proof of the optimal sphere-packing density in dimension 24, Cohn, Kumar, Miller, Radchenko, and Viazovska (CKMRV) established that the Leech lattice $\Lambda_{24}$ is the unique periodic configuration attaining density $\frac{\pi^{12}}{12!}$. The uniqueness argument in their Section 7 contains an elegant structural mechanism: starting from an arbitrary optimal periodic packing, one forms the $\mathbb{Z}$-span $M = \mathrm{span}_{\mathbb{Z}}(X)$ of its centers and proves that $M$ is an even unimodular lattice with no vectors of squared norm 2.

This note gives a detailed expository reconstruction of that argument, conditional on the CKMRV auxiliary-function construction, the dimension-24 sphere-packing bound, and the Niemeier classification. We derive the equality conditions in the periodic Poisson summation formula, establish the lattice inclusions $L \subseteq M \subseteq L^*$, and explain how the Fourier structure factor excludes roots of $M$. In particular, if $\mathbf{r} \in M$ had squared norm 2, the Fourier equality condition would force $S(\mathbf{r}) = 0$, whereas integrality of $M$ forces $S(\mathbf{r}) = m^2 > 0$, a contradiction.

We also present an explicit five-point configuration in $D_4$ illustrating why pairwise distance constraints alone do not prevent the integer span of a point set from containing a root. Finally, a covolume squeeze yields $\mathrm{Vol}(M) = 1$ and $[M : L] = m$, after which the Niemeier classification identifies $M \cong \Lambda_{24}$ and coset exhaustion proves $X = M$. Thus the argument explains explicitly how the lattice structure emerges intrinsically from an arbitrary optimal periodic configuration.

## 1 Introduction

In 2017, Cohn, Kumar, Miller, Radchenko, and Viazovska [2] completed the breakthrough proof that the maximum sphere-packing density in $\mathbb{R}^{24}$ is $\frac{\pi^{12}}{12!}$, saturated uniquely by the Leech lattice $\Lambda_{24}$. While their universal upper bound has received extensive exposition, the uniqueness argument for periodic configurations—presented compactly in Section 7 of [2]—contains several subtle structural mechanisms that warrant a line-by-line examination.

The goal of this note is purely expository:

1. To provide a fully expanded, self-contained derivation of the periodic uniqueness argument of [2], explicitly identifying all imported theorems.

2. To clarify why the integer span $M = \mathrm{span}_{\mathbb{Z}}(X)$ forms an even integral lattice satisfying the fundamental inclusion $L \subseteq M \subseteq L^*$.

3. To exhibit an explicit five-point configuration in $D_4$ showing that a generating set can satisfy the pairwise exclusion condition $\|\mathbf{u} - \mathbf{v}\|^2 \geq 4$ everywhere while its integer span

contains root vectors of squared norm 2, demonstrating why real-space distance geometry alone cannot prove root-exclusion without the Fourier structure factor.

4. To emphasize that the resulting theorem establishes that $X$ is congruent to $\Lambda_{24}$ as an intrinsic point set, while the initial period lattice $L$ is a sub-lattice of index $m$.

Let $X \subset \mathbb{R}^{24}$ be a periodic configuration of centers of open unit balls ($R = 1$). The configuration is expressed as a finite union of translates of a full-rank period lattice $L \subset \mathbb{R}^{24}$:

$$X = \bigcup_{j=1}^{m} (\mathbf{x}_j + L), \qquad \mathbf{x}_j \in \mathbb{R}^{24}, \tag{1}$$

where the representatives $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are pairwise distinct modulo $L$.

The non-overlapping packing condition is

$$\|\mathbf{u} - \mathbf{v}\| \geq 2 \qquad \text{for all distinct } \mathbf{u}, \mathbf{v} \in X. \tag{2}$$

The packing density $\delta(X)$ is defined by

$$\delta(X) = \frac{m \ \mathrm{Vol}(B_{24}(1))}{\mathrm{Vol}(L)} = \frac{m}{\mathrm{Vol}(L)} \frac{\pi^{12}}{12!}, \tag{3}$$

where $\mathrm{Vol}(B_{24}(1)) = \frac{\pi^{12}}{12!}$ is the Euclidean volume of the unit ball in $\mathbb{R}^{24}$.

The sphere-packing theorem of CKMRV [2] establishes that

$$\delta(X) \leq \frac{\pi^{12}}{12!} \tag{4}$$

with equality attained by the Leech lattice $\Lambda_{24}$. Our purpose is to reconstruct uniqueness among periodic configurations directly from the equality conditions.

**Theorem 1.1** (Uniqueness of Optimal Periodic Packings [2])**.** *Let $X = \bigcup_{j=1}^{m} (\mathbf{x}_j + L) \subset \mathbb{R}^{24}$ be a periodic packing of unit spheres achieving density*

$$\delta(X) = \frac{\pi^{12}}{12!}.$$

*Then $X$ is congruent to the Leech lattice $\Lambda_{24}$.*

# 2 Equality Conditions and Covolume of $L$

We import the radial auxiliary function constructed by CKMRV [2].

**Theorem 2.1** (Imported Auxiliary Function [2])**.** *There exists a radial Schwartz function $f \in \mathcal{S}(\mathbb{R}^{24})$ such that:*

1. *Normalization: $f(\mathbf{0}) = \widehat{f}(\mathbf{0}) = 1$;*

2. *Real Sign Condition: $f(\mathbf{x}) \leq 0$ for $\|\mathbf{x}\| \geq 2$;*

3. *Dual Positivity: $\widehat{f}(\mathbf{k}) \geq 0$ for all $\mathbf{k} \in \mathbb{R}^{24}$;*

4. *Real Zeros: On $\|\mathbf{x}\| \geq 2$, $f(\mathbf{x}) = 0 \iff \|\mathbf{x}\|^2 \in \{4, 6, 8, \ldots\}$;*

5. *Dual Zeros: For $\mathbf{k} \neq \mathbf{0}$, $\widehat{f}(\mathbf{k}) = 0 \iff \|\mathbf{k}\|^2 \in \{4, 6, 8, \ldots\}$. In particular, $\widehat{f}(\mathbf{k}) > 0$ whenever $0 < \|\mathbf{k}\|^2 < 4$.*

We adopt the standard Fourier transform convention:

$$\widehat{f}(\mathbf{k}) = \int_{\mathbb{R}^{24}} f(\mathbf{x}) e^{-2\pi i \langle \mathbf{x}, \mathbf{k} \rangle} \, d\mathbf{x}. \tag{5}$$

For the periodic configuration $X$, the Poisson summation formula yields:

$$\sum_{j,\ell=1}^{m} \sum_{\mathbf{x} \in L} f(\mathbf{x}_j - \mathbf{x}_\ell + \mathbf{x}) = \frac{1}{\mathrm{Vol}(L)} \sum_{\mathbf{k} \in L^*} S(\mathbf{k}) \widehat{f}(\mathbf{k}), \tag{6}$$

where $L^*$ is the reciprocal lattice of $L$ ($\mathrm{Vol}(L^*) = \frac{1}{\mathrm{Vol}(L)}$), and $S(\mathbf{k})$ is the dual structure factor:

$$S(\mathbf{k}) = \left| \sum_{j=1}^{m} e^{2\pi i \langle \mathbf{k}, \mathbf{x}_j \rangle} \right|^2 \geq 0, \qquad S(\mathbf{0}) = m^2. \tag{7}$$

**Lemma 2.2** (Covolume of the Period Lattice)**.** *If $X$ attains the optimal density $\delta(X) = \frac{\pi^{12}}{12!}$, then*

$$\mathrm{Vol}(L) = m \qquad \textit{and} \qquad \mathrm{Vol}(L^*) = \frac{1}{m}.$$

*Proof.* By the packing condition (2), the only terms on the left-hand side of (6) with $\|\mathbf{x}_j - \mathbf{x}_\ell + \mathbf{x}\| < 2$ are the $m$ diagonal terms where $j = \ell$ and $\mathbf{x} = \mathbf{0}$, each contributing $f(\mathbf{0}) = 1$. Every other displacement has norm $\geq 2$, where $f \leq 0$. Therefore,

$$\sum_{j,\ell=1}^{m} \sum_{\mathbf{x} \in L} f(\mathbf{x}_j - \mathbf{x}_\ell + \mathbf{x}) \leq m. \tag{8}$$

On the Fourier side, all summands are non-negative. Isolating $\mathbf{k} = \mathbf{0}$ gives:

$$\frac{m^2}{\mathrm{Vol}(L)} \leq \frac{1}{\mathrm{Vol}(L)} \sum_{\mathbf{k} \in L^*} S(\mathbf{k}) \widehat{f}(\mathbf{k}). \tag{9}$$

Combining (8) and (9) yields $\frac{m^2}{\mathrm{Vol}(L)} \leq m$, whence $\mathrm{Vol}(L) \geq m$.

On the other hand, the density is $\delta(X) = \frac{m}{\mathrm{Vol}(L)} \frac{\pi^{12}}{12!} = \frac{\pi^{12}}{12!}$, which forces $\frac{m}{\mathrm{Vol}(L)} = 1$. Therefore, $\mathrm{Vol}(L) = m$ and $\mathrm{Vol}(L^*) = \frac{1}{m}$. □

**Corollary 2.3** (Exact Equality Vanishing Conditions)**.** *For every non-zero displacement $\mathbf{u} - \mathbf{v} \in X - X$,*

$$\|\mathbf{u} - \mathbf{v}\|^2 \in \{4, 6, 8, \dots\} \subset 2\mathbb{Z}_{\geq 2}. \tag{10}$$

*Moreover, the dual vanishing condition holds:*

$$S(\mathbf{k}) \widehat{f}(\mathbf{k}) = 0 \qquad \textit{for every } \mathbf{k} \in L^* \setminus \{\mathbf{0}\}. \tag{11}$$

*Proof.* In Lemma 2.2, $\frac{m^2}{\mathrm{Vol}(L)} = m$, so equality holds throughout (6). Consequently, every non-zero real displacement must evaluate to a zero of $f$, which by Theorem 2.1 gives the displacement spectrum. Likewise, every non-zero Fourier contribution must vanish, establishing (11). □

# 3 The Primal Span and the Backbone $L \subseteq M \subseteq L^*$

Without loss of generality, we translate the configuration $X$ so that $\mathbf{x}_1 = \mathbf{0} \in X$.

**Definition 3.1** (The Primal Span Lattice $M$)**.** We define $M$ as the $\mathbb{Z}$-linear span of the point configuration:

$$M := \operatorname{span}_{\mathbb{Z}}(X) = L + \sum_{j=2}^{m} \mathbb{Z}\mathbf{x}_j \subset \mathbb{R}^{24}. \tag{12}$$

**Lemma 3.2** (Dual Inclusion)**.** *The period lattice $L$ is even, every point $\mathbf{x}_j \in X$ belongs to the reciprocal lattice $L^*$, and the primal span satisfies:*

$$\boxed{L \subseteq M \subseteq L^*}. \tag{13}$$

*Proof.* Since $\mathbf{0} \in X$, for any $\boldsymbol{\ell} \in L \setminus \{\mathbf{0}\}$, $\boldsymbol{\ell} = \mathbf{0} - (-\boldsymbol{\ell}) \in X - X$. Corollary 2.3 implies $\|\boldsymbol{\ell}\|^2 \in \{4, 6, 8, \dots\} \subset 2\mathbb{Z}$. Thus $L$ is an even lattice, so $\langle \boldsymbol{\ell}_1, \boldsymbol{\ell}_2 \rangle = \frac{1}{2}(\|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2\|^2 - \|\boldsymbol{\ell}_1\|^2 - \|\boldsymbol{\ell}_2\|^2) \in \mathbb{Z}$, which gives $L \subseteq L^*$.

Next, fix $j \in \{1, \dots, m\}$ and $\boldsymbol{\ell} \in L$. Both $\mathbf{x}_j$ and $\mathbf{x}_j + \boldsymbol{\ell}$ belong to $X$. Since $\mathbf{0} \in X$, Corollary 2.3 implies $\|\mathbf{x}_j\|^2 \in 2\mathbb{Z}$ and $\|\mathbf{x}_j + \boldsymbol{\ell}\|^2 \in 2\mathbb{Z}$. When $\mathbf{x}_j = \mathbf{0}$, $\langle \mathbf{x}_j, \boldsymbol{\ell} \rangle = 0 \in \mathbb{Z}$ trivially. When $\mathbf{x}_j \neq \mathbf{0}$ and $\boldsymbol{\ell} \neq \mathbf{0}$, polarization yields:

$$2\langle \mathbf{x}_j, \boldsymbol{\ell} \rangle = \|\mathbf{x}_j + \boldsymbol{\ell}\|^2 - \|\mathbf{x}_j\|^2 - \|\boldsymbol{\ell}\|^2 \in 2\mathbb{Z} - 2\mathbb{Z} - 2\mathbb{Z} = 2\mathbb{Z}. \tag{14}$$

Dividing by 2 gives $\langle \mathbf{x}_j, \boldsymbol{\ell} \rangle \in \mathbb{Z}$ for all $\boldsymbol{\ell} \in L$. By definition of the reciprocal lattice $L^* := \{\mathbf{y} \in \mathbb{R}^{24} : \langle \mathbf{y}, \boldsymbol{\ell} \rangle \in \mathbb{Z}\ \forall \boldsymbol{\ell} \in L\}$, this establishes that $\mathbf{x}_j \in L^*$ for each $j \in \{1, \dots, m\}$.

Since $L \subseteq L^*$ and every $\mathbf{x}_j \in L^*$, every integer linear combination of these generators lies in $L^*$. Thus $L \subseteq M \subseteq L^*$. □

**Theorem 3.3** (Even Integrality of $M$)**.** *The lattice $M$ is an even integral lattice of rank* 24.

*Proof.* For any two generators $\mathbf{u}, \mathbf{v} \in X$, both $\|\mathbf{u}\|^2$ and $\|\mathbf{v}\|^2$ are even integers (since $\mathbf{0} \in X$), and $\|\mathbf{u} - \mathbf{v}\|^2 \in 2\mathbb{Z}$ by Corollary 2.3. Polarization gives:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2\right) \in \mathbb{Z}. \tag{15}$$

Thus, all entries of the Gram matrix of generators are integers. For any arbitrary element $\mathbf{w} = \sum_i a_i \mathbf{u}_i \in M$ with $a_i \in \mathbb{Z}$ and $\mathbf{u}_i \in X$, the squared norm expands as:

$$\|\mathbf{w}\|^2 = \sum_i a_i^2 \|\mathbf{u}_i\|^2 + 2\sum_{i<j} a_i a_j \langle \mathbf{u}_i, \mathbf{u}_j \rangle \in 2\mathbb{Z}. \tag{16}$$

Furthermore, for any $\mathbf{w}_1, \mathbf{w}_2 \in M$, bilinearity forces $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle \in \mathbb{Z}$. Since $L \subseteq M$ and $\operatorname{rank}(L) = 24$, $M$ is an even integral lattice of rank 24. □

# 4 An Explicit $D_4$ Counterexample to Real-Space Root Exclusion

It is tempting to conjecture that because all pairwise differences in $X$ satisfy $\|\mathbf{u} - \mathbf{v}\|^2 \geq 4$, the integer span $M = \operatorname{span}_{\mathbb{Z}}(X)$ must automatically have minimum non-zero squared norm $\geq 4$.

However, an arbitrary integer combination $\mathbf{w} = \sum a_i \mathbf{u}_i$ is not generally a pairwise difference $\mathbf{u} - \mathbf{v}$. The following construction is not a periodic packing; it is an algebraic counterexample illustrating why pairwise distance constraints on a generating set do not imply a minimum-norm bound on its $\mathbb{Z}$-span:

**Example 4.1** (A Root-Generating Configuration in $D_4$)**.** Consider the standard root lattice $D_4 \subset \mathbb{Z}^4$. Define the five-point configuration $X = \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\} \subset D_4$ by:

$$\mathbf{x}_0 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix}-1\\2\\1\\0\end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix}0\\-2\\0\\2\end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix}-1\\0\\-1\\-2\end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix}1\\2\\0\\1\end{pmatrix}.$$

The exact $5 \times 5$ Gram matrix $G_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle$ is:

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & -4 & 0 & 3 \\ 0 & -4 & 8 & -4 & -2 \\ 0 & 0 & -4 & 6 & -3 \\ 0 & 3 & -2 & -3 & 6 \end{pmatrix}. \tag{17}$$

The resulting $5 \times 5$ squared distance matrix $D^2_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|^2 = G_{ii} + G_{jj} - 2G_{ij}$ is:

$$D^2 = \begin{pmatrix} 0 & 6 & 8 & 6 & 6 \\ 6 & 0 & 22 & 12 & 6 \\ 8 & 22 & 0 & 22 & 18 \\ 6 & 12 & 22 & 0 & 18 \\ 6 & 6 & 18 & 18 & 0 \end{pmatrix}. \tag{18}$$

Every non-zero pairwise squared distance satisfies:

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 \in \{6, 8, 12, 18, 22\} \subset 2\mathbb{Z}_{\geq 2}. \tag{19}$$

No two points in $X$ have squared distance 2. Nevertheless, consider the integer linear combination:

$$\mathbf{r} := \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 = \begin{pmatrix}0\\-2\\0\\2\end{pmatrix} + \begin{pmatrix}-1\\0\\-1\\-2\end{pmatrix} + \begin{pmatrix}1\\2\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\-1\\1\end{pmatrix}. \tag{20}$$

The vector $\mathbf{r} \in \mathrm{span}_{\mathbb{Z}}(X)$ has squared norm:

$$\|\mathbf{r}\|^2 = 0^2 + 0^2 + (-1)^2 + 1^2 = \mathbf{2}. \tag{21}$$

Thus, the integer span $\mathrm{span}_{\mathbb{Z}}(X)$ contains a root vector, despite $X$ strictly satisfying the pairwise exclusion condition $\|\mathbf{u} - \mathbf{v}\|^2 \geq 4$.

Example 4.1 shows that root exclusion cannot be derived from real-space exclusions alone. It requires the dual Fourier equality equations.

# 5 Root Exclusion via the Structure Factor (The CKMRV Argument)

We now isolate the root-exclusion step of the CKMRV argument [2]:

**Theorem 5.1** (Root Exclusion in $M$ [2])**.** *The lattice $M$ contains no vector of squared norm 2:*

$$\boxed{M \cap \{\mathbf{r} \in \mathbb{R}^{24} : \|\mathbf{r}\|^2 = 2\} = \varnothing}. \tag{22}$$

*Proof.* Suppose, for the sake of contradiction, that there exists a vector $\mathbf{r} \in M$ with $\|\mathbf{r}\|^2 = 2$.

By Lemma 3.2, $M \subseteq L^*$. Therefore:

$$\mathbf{r} \in L^* \setminus \{\mathbf{0}\}. \tag{23}$$

The dual Fourier equality condition (11) of Corollary 2.3 evaluated at $\mathbf{k} = \mathbf{r}$ requires:

$$S(\mathbf{r})\widehat{f}(\mathbf{r}) = 0. \tag{24}$$

Since $\|\mathbf{r}\|^2 = 2$, Theorem 2.1 (item 5) implies that $\mathbf{r}$ is not a zero of $\widehat{f}$. Because $\widehat{f} \geq 0$ everywhere, it follows that $\widehat{f}(\mathbf{r}) > 0$. Consequently, equation (24) forces:

$$S(\mathbf{r}) = 0. \tag{25}$$

On the other hand, we compute $S(\mathbf{r})$ directly from its definition. By Theorem 3.3, $M$ is an integral lattice. Since $\mathbf{r} \in M$ and each representative $\mathbf{x}_j \in X \subseteq M$, the inner product between them is strictly an integer:

$$\langle \mathbf{r}, \mathbf{x}_j \rangle \in \mathbb{Z} \qquad \text{for every } j \in \{1, \ldots, m\}. \tag{26}$$

Therefore, every phase factor in the structure factor evaluates to unity:

$$e^{2\pi i \langle \mathbf{r}, \mathbf{x}_j \rangle} = 1 \qquad \text{for every } j \in \{1, \ldots, m\}. \tag{27}$$

Using the definition of the structure factor (7):

$$S(\mathbf{r}) = \left| \sum_{j=1}^{m} e^{2\pi i \langle \mathbf{r}, \mathbf{x}_j \rangle} \right|^2 = \left| \sum_{j=1}^{m} 1 \right|^2 = m^2. \tag{28}$$

Because $m \geq 1$, $S(\mathbf{r}) = m^2 \geq 1 > 0$.

Equations (25) and (28) contradict each other ($0 = m^2 \geq 1$). Hence, no such vector $\mathbf{r}$ can exist in $M$. □

**Corollary 5.2** (Rootless Primal Lattice)**.** *The lattice $M$ satisfies:*

$$\min_{\mathbf{w} \in M \setminus \{\mathbf{0}\}} \|\mathbf{w}\|^2 \geq 4. \tag{29}$$

*Proof.* By Theorem 3.3, $M$ is an even integral lattice, so its squared norms are even integers. The hard-core exclusion condition rules out $\|\mathbf{w}\|^2 = 0$, and Theorem 5.1 rules out $\|\mathbf{w}\|^2 = 2$. Thus $\min_{\mathbf{w} \neq \mathbf{0}} \|\mathbf{w}\|^2 \geq 4$. □

## 6 The Covolume Squeeze

**Lemma 6.1** (Coset Index Upper Bound)**.** *The inclusion $L \subseteq M$ satisfies $[M : L] \geq m$, and consequently:*

$$\mathrm{Vol}(M) \leq 1. \tag{30}$$

*Proof.* The representatives $\mathbf{x}_1, \ldots, \mathbf{x}_m$ define $m$ pairwise disjoint cosets of $L$. Since $X \subset M$, all $m$ disjoint cosets $\mathbf{x}_j + L$ are contained in $M$. Therefore, $[M : L] \geq m$. Using $\mathrm{Vol}(L) = m$ from Lemma 2.2:

$$\mathrm{Vol}(M) = \frac{\mathrm{Vol}(L)}{[M : L]} \leq \frac{m}{m} = 1. \tag{31}$$

□

**Lemma 6.2** (Covolume Bound from the Dimension-24 Packing Theorem)**.** *The covolume of $M$ satisfies:*

$$\mathrm{Vol}(M) \geq 1. \tag{32}$$

*Proof.* By Corollary 5.2, $M$ is a full-rank lattice in $\mathbb{R}^{24}$ with minimum squared norm at least 4. Open balls of radius 1 centered at points of $M$ form a non-overlapping sphere packing. By the established dimension-24 sphere-packing theorem of CKMRV [2], no sphere packing of unit balls in $\mathbb{R}^{24}$ can exceed density $\frac{\pi^{12}}{12!}$. Applying this bound directly to the lattice packing $M$ gives:

$$\frac{\mathrm{Vol}(B_{24}(1))}{\mathrm{Vol}(M)} \leq \frac{\pi^{12}}{12!} = \mathrm{Vol}(B_{24}(1)). \tag{33}$$

Dividing both sides by $\mathrm{Vol}(B_{24}(1))$ yields:

$$\mathrm{Vol}(M) \geq 1. \tag{34}$$

□

**Theorem 6.3** (Covolume Squeeze on $M$)**.** *The lattice $M$ is unimodular:*

$$\boxed{\mathrm{Vol}(M) = 1 \qquad \textit{and} \qquad [M : L] = m}. \tag{35}$$

*Proof.* Combining (31) and (34) yields $1 \leq \mathrm{Vol}(M) \leq 1$, so $\mathrm{Vol}(M) = 1$. Substituting this into $\mathrm{Vol}(M) = \frac{m}{[M:L]}$ forces $[M : L] = m$. □

# 7 Identification of the Lattice with $\Lambda_{24}$

**Theorem 7.1** (Unimodular Rootless Structure)**.** *The lattice $M$ is isometric to the Leech lattice $\Lambda_{24}$.*

*Proof.* By Theorem 3.3, $M$ is an even integral lattice of rank 24. By Theorem 6.3, it has covolume 1, so it is an even unimodular lattice. Moreover, Corollary 5.2 establishes that $\min_{\mathbf{w}\neq\mathbf{0}} \|\mathbf{w}\|^2 \geq 4$, so $M$ contains no roots of squared norm 2.

By the classification of positive-definite even unimodular lattices of rank 24 (Niemeier [3]), there exist exactly 24 isometry classes, and the Leech lattice $\Lambda_{24}$ is the unique class with no roots. Therefore:

$$M \cong \Lambda_{24}. \tag{36}$$

□

# 8 Coset Exhaustion and Main Theorem

We now identify the periodic configuration $X$ with the lattice $M$.

**Lemma 8.1** (Coset Exhaustion)**.** *Let $L \subseteq M$ be lattices of full rank in $\mathbb{R}^{24}$, and suppose $[M : L] = m$. If $\mathbf{x}_1, \ldots, \mathbf{x}_m$ represent distinct cosets of $L$ in $M$, then*

$$M = \bigcup_{j=1}^{m} (\mathbf{x}_j + L). \tag{37}$$

*Proof.* The quotient group $M/L$ has order $[M : L] = m$. Since the representatives $\mathbf{x}_1, \ldots, \mathbf{x}_m$ define $m$ pairwise disjoint cosets of $L$ contained in $M$, they form a complete system of coset representatives for $M/L$. Their union therefore exhausts $M$ completely. □

**Theorem 8.2** (Proof of Theorem 1.1)**.** *Let $X = \bigcup_{j=1}^{m}(\mathbf{x}_j + L) \subset \mathbb{R}^{24}$ be a periodic sphere packing achieving the Cohn–Elkies upper bound $\delta(X) = \frac{\pi^{12}}{12!}$. Then $X$ is congruent to $\Lambda_{24}$.*

*Proof.* By definition, $X = \bigcup_{j=1}^{m}(\mathbf{x}_j + L)$. Since $L \subseteq M$ and every representative $\mathbf{x}_j$ belongs to $M$ by construction, each coset $\mathbf{x}_j + L$ is contained in $M$. Thus $X \subseteq M$.

By Theorem 6.3, the index satisfies $[M : L] = m$. Because the $m$ points $\mathbf{x}_1, \dots, \mathbf{x}_m$ are pairwise distinct modulo $L$, Lemma 8.1 establishes:

$$X = \bigcup_{j=1}^{m}(\mathbf{x}_j + L) \equiv M. \tag{38}$$

By Theorem 7.1, $M \cong \Lambda_{24}$. Therefore, $X$ is congruent to $\Lambda_{24}$. □

# 9 Remarks on the Period Lattice

The conclusion of Theorem 8.2 concerns the intrinsic center configuration $X$, not the particular period lattice $L$ appearing in an arbitrary periodic presentation.

Once $X = \Lambda_{24}$, any finite-index sublattice $L \subseteq \Lambda_{24}$ can serve as a period lattice, provided the chosen representatives $\mathbf{x}_1, \dots, \mathbf{x}_m$ form a complete system of coset representatives for $\Lambda_{24}/L$. In that presentation:

$$m = [\Lambda_{24} : L] \qquad \text{and} \qquad \mathrm{Vol}(L) = m\,\mathrm{Vol}(\Lambda_{24}) = m. \tag{39}$$

Thus, the periodic presentation need not have $m = 1$ relative to an arbitrarily chosen period lattice $L$. Rather, the point set $X$ itself is an additive lattice congruent to $\Lambda_{24}$, which possesses a primitive period representation with $m = 1$.

# 10 Conclusion

Starting from an arbitrary periodic configuration, we form the $\mathbb{Z}$-span of its centers, $M = \mathrm{span}_{\mathbb{Z}}(X)$. The equality conditions force this span to be an even unimodular rootless lattice, and the coset-exhaustion argument then shows that the original configuration coincides with that lattice:

$$\boxed{\begin{aligned}
\text{Optimal Density } \delta(X) = \tfrac{\pi^{12}}{12!} \quad &\Longrightarrow \quad \mathrm{Vol}(L) = m \quad \text{and} \quad \|\mathbf{u} - \mathbf{v}\|^2 \in 2\mathbb{Z}_{\geq 2} \\
&\Longrightarrow \quad L \subseteq M \subseteq L^* \\
&\Longrightarrow \quad \text{Fourier root exclusion via } S(\mathbf{r}) = 0 \text{ vs. } S(\mathbf{r}) = m^2 \\
&\Longrightarrow \quad \min(M \setminus \{\mathbf{0}\})^2 \geq 4 \implies \mathrm{Vol}(M) \geq 1 \\
&\Longrightarrow \quad [M : L] \geq m \implies \mathrm{Vol}(M) \leq 1 \\
&\Longrightarrow \quad \mathrm{Vol}(M) = 1 \quad \text{and} \quad [M : L] = m \\
&\Longrightarrow \quad M \cong \Lambda_{24} \quad \text{(Niemeier 1973)} \\
&\Longrightarrow \quad X = M \cong \Lambda_{24} \quad \text{(Coset Exhaustion).}
\end{aligned}} \tag{40}$$

The essential point is that no lattice hypothesis is imposed on the original packing. The lattice emerges intrinsically through the $\mathbb{Z}$-span of the centers. The Fourier structure factor excludes roots in that span, while the primal and dual covolume estimates force unimodularity, locking the configuration uniquely into the Leech lattice.

## References

[1] H. Cohn and N. Elkies, *New upper bounds on sphere packings I*, Ann. of Math. (2) **157** (2003), no. 2, 689–714.

[2] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska, *The sphere packing problem in dimension 24*, Ann. of Math. (2) **185** (2017), no. 3, 1017–1033.

[3] H.-V. Niemeier, *Definite quadratische Formen der Dimension 24 und Diskriminate 1*, J. Number Theory **5** (1973), 142–178.